VISIT…

SECTION 23

LINKAGE

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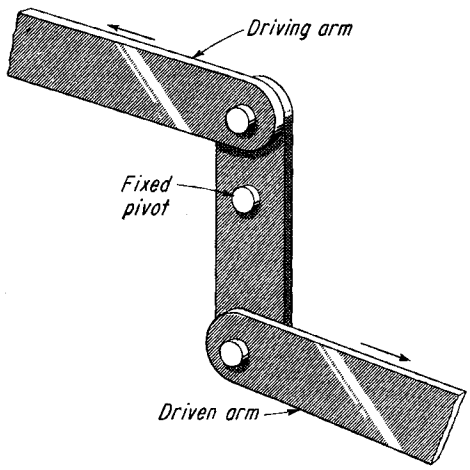
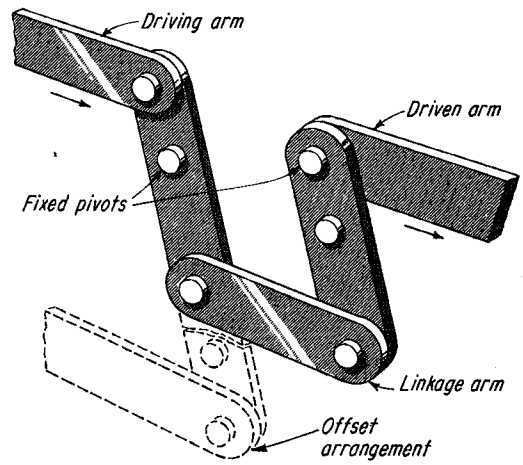
8 Basic Push-Pull Linkages

These arrangements are invariably the root of all linkage devices.

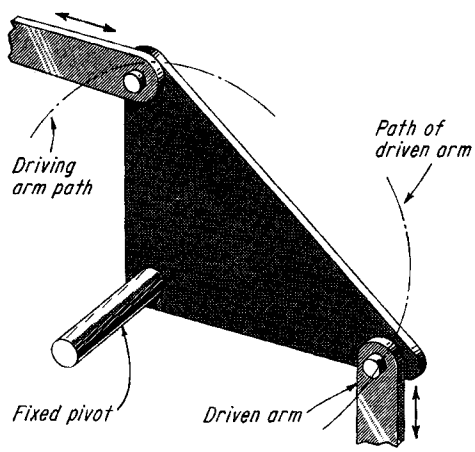
Frank William Wood JR.



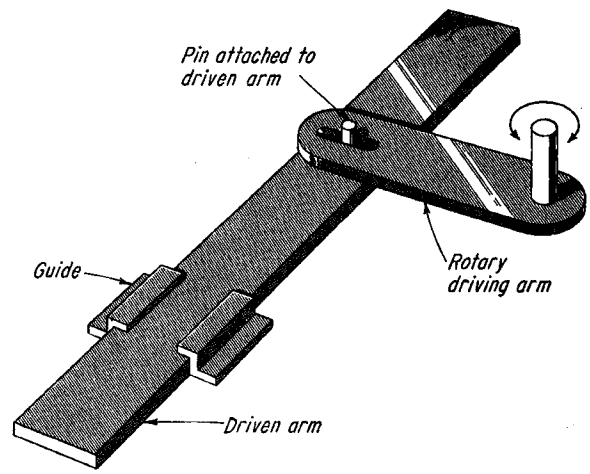
1 **FIXED PIVOTS** on arm lengths are located to control ratio of input and output movements of this push-pull-actuated linkage. Mechanism can be either flat bars or round rods of adequate thickness to prevent bowing under compression.



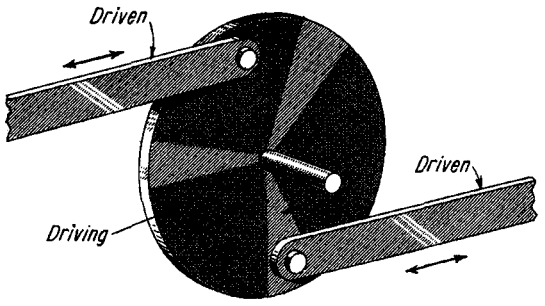
2 **PUSH-PULL LINKAGE** for same direction of motion can be obtained by adding linkage arm to previous design. In both cases, if arms are bars it might be best to make them forked rather than merely flatted at their linkage ends.



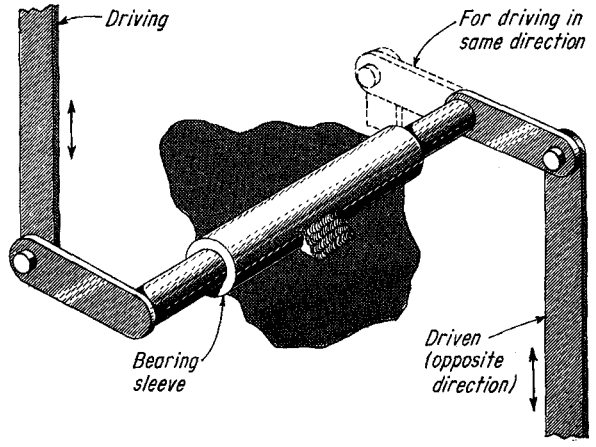
5 **VERTICAL OUTPUT MOVEMENT** from horizontal input is gained with this push-pull linkage. Although the triangular-shaped plate could be substituted by an L-shaped arm, the plate gives greater freedom of driving- and driven-arm location.



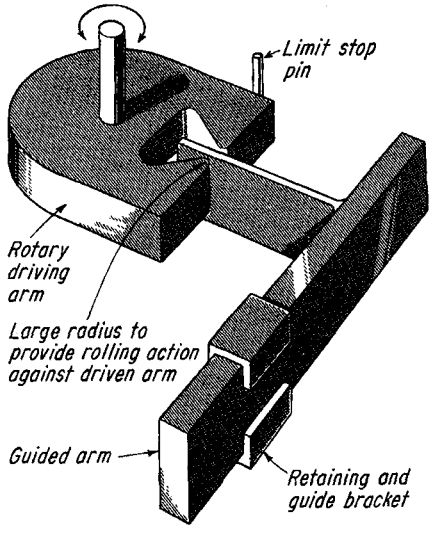
6 **FOR LIMITED STRAIGHT-LINE** 2-direction motion use this rotary-actuated linkage. Friction between the pin and sides of slot limit this design to small loads. A bearing on the pin will reduce friction and slot wear to negligible proportions.



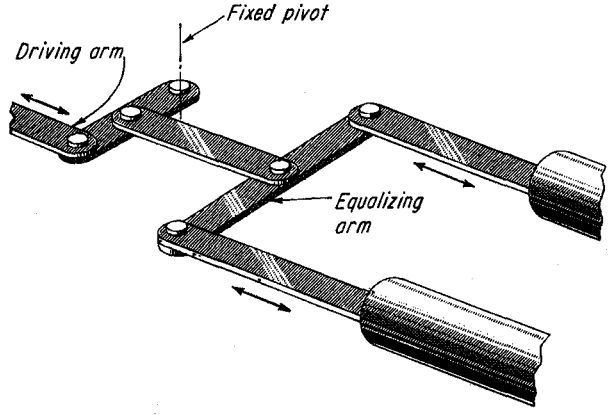
3 **ROTARY-ACTUATED LINKAGE** gives opposite direction of motion and can be obtained by using 3-bar linkage with pivot point of middle link located at midpoint of arm length. Disk should be adequately strengthened for heavy loads.



4 **SAME-DIRECTION MOTION** is given by this rotary-actuated linkage when end arms are located on the same sides; for opposite-direction motion, locate the arms on opposite sides. Use when a crossover is required between input and output.


Limit stop pin
Rotary driving arm
Large radius to provide rolling action against driven arm
Guided arm
Retaining and guide bracket

7 **THIS ROTARY-ACTUATED** linkage for straight-line 2-direction motion has rotary driving arm with a modified dovetail opening that fits freely around a flat sheet or bar arm. Driven arm reciprocates in slot as rotary driving arm is turned.

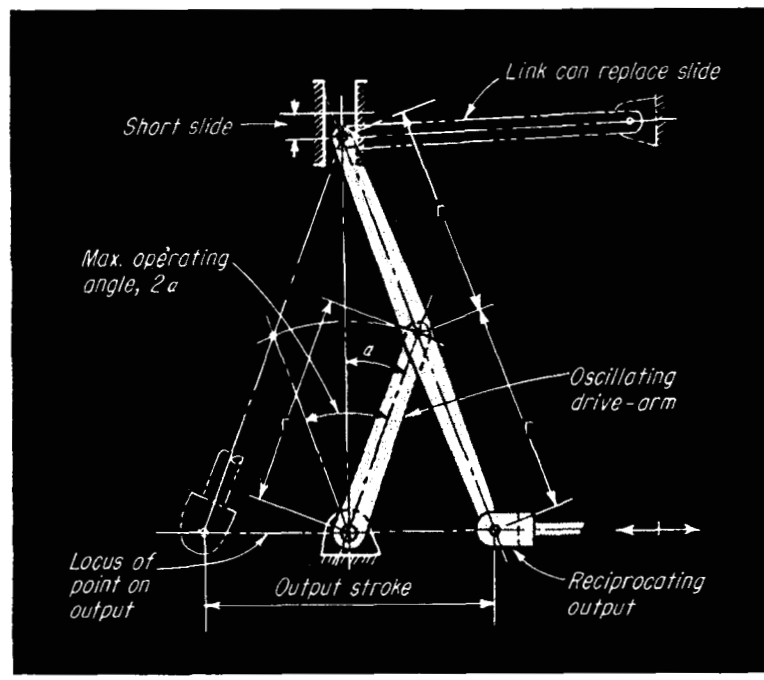


8 **EQUALIZING LINKAGE** here has an equalizing arm that balances the input force to two output arms. This arrangement is most suitable for air or hydraulic systems where equal force is to be exerted on the pistons of separate cylinders.

5 Linkages for Straight-Line Motion

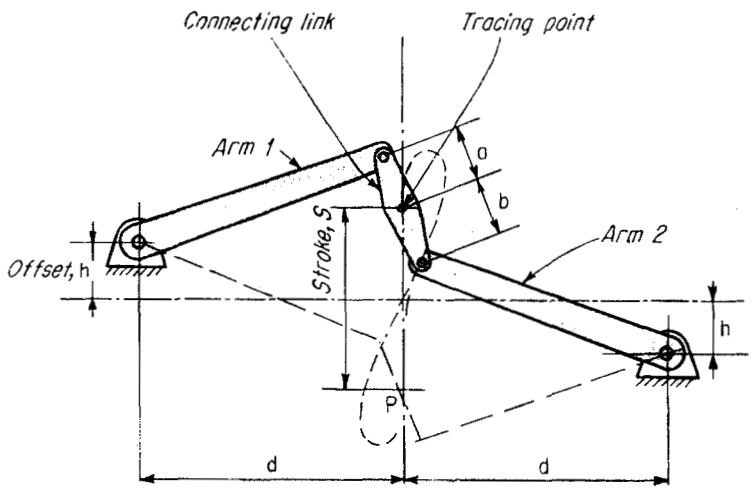
These devices convert rotary to straight-line motion without the need for guides.

Sigmund Rappaport



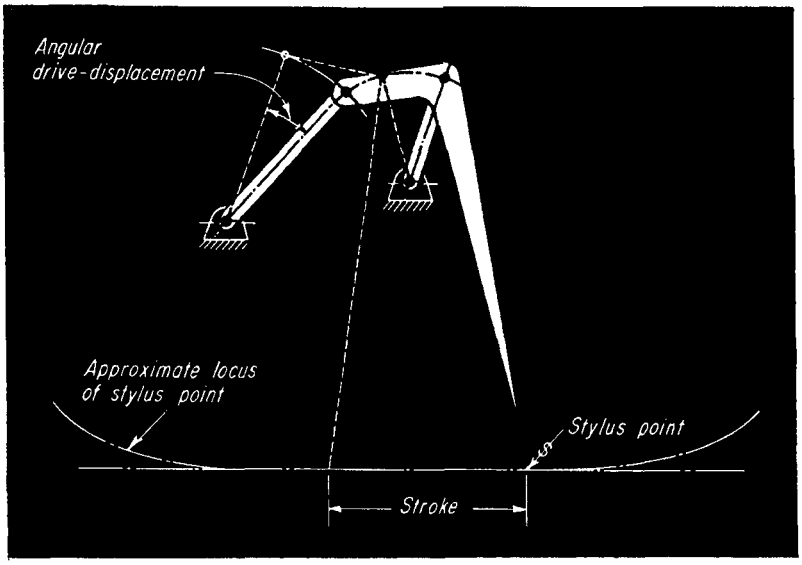
1

Evans' linkage . . .
has oscillating drive-arm that should have a maximum operating angle of about 40°. For a relatively short guide-way, the reciprocating output stroke is large. Output motion is on a true straight line in true harmonic motion. If an exact straight-line motion is not required, however, a link can replace the slide. The longer this link, the closer does the output motion approach that of a true straight line—if link-length equals output stroke, deviation from straight-line motion is only 0.03% of output stroke.



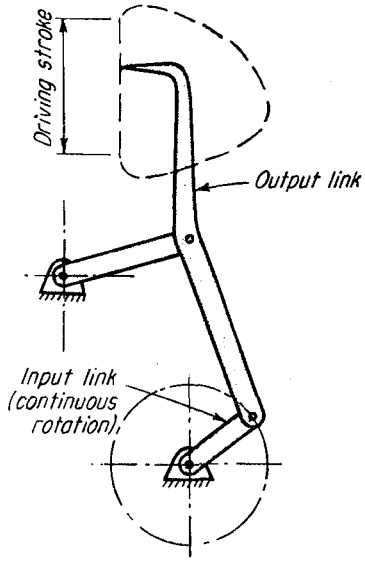
2

Simplified Watt's linkage . . .
generates an approximate straight-line motion. If the two arms are equally long, the tracing point describes a symmetrical figure 8 with an almost straight line throughout the stroke length. The straightest and longest stroke occurs when the connecting-link length is about ⅔ of the stroke, and arm length is 1.5S. Offset should equal half the connecting-link length. If the arms are unequal, one branch of the figure-8 curve is straighter than the other. It is straightest when a/b equals (arm 2)/(arm 1).



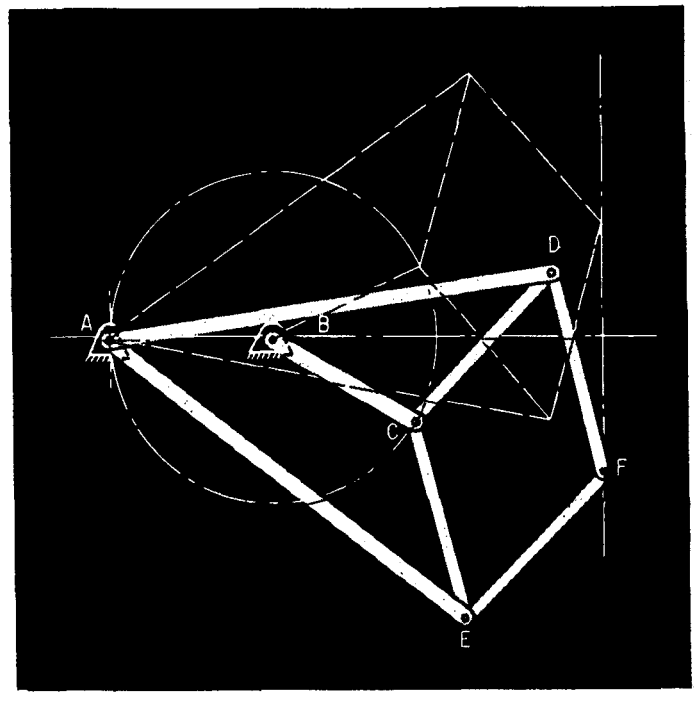
3

Four-bar linkage . . .
produces approximately straight-line motion. This arrangement provides motion for the stylus on self-registering measuring instruments. A comparatively small drive-displacement results in a long, almost-straight line.



4

D-drive . . .
results when linkage arms are arranged as shown here. Output-link point describes a path resembling the letter *D*, thus it contains a straight portion as part of its cycle. Motion is ideal for quick engagement and disengagement before and after a straight driving-stroke. Example, the intermittent film-drive in movie-film projectors.



5

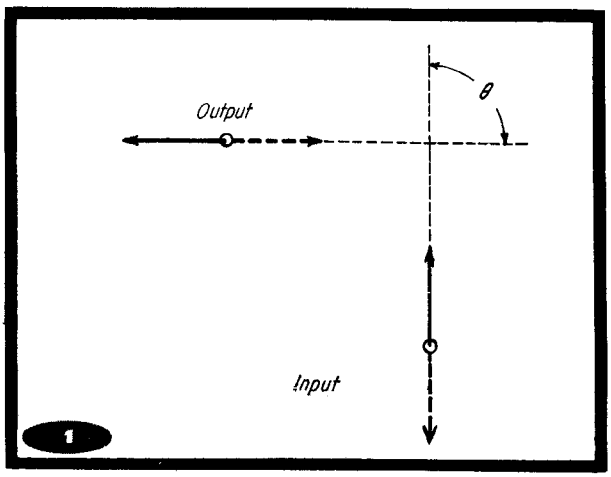
The "Peaucellier cell" . . .
was first solution to the classical problem of generating a straight line with a linkage. Its basis: within the physical limits of the motion, AC x *AF* remains constant. Curves described by C and *F* are, therefore, inverse; if C describes a circle that goes through *A*, then *F* will describe a circle of infinite radius—a straight line, perpendicular to *AB*. The only requirements are: $AB=BC$; $AD=AE$; and *CD*, *DF*, *FE*, *EC* are all equal. The linkage can be used to generate circular arcs of large radius by locating *A* outside the circular path of C.

10 Ways to Change Straight-Line Direction

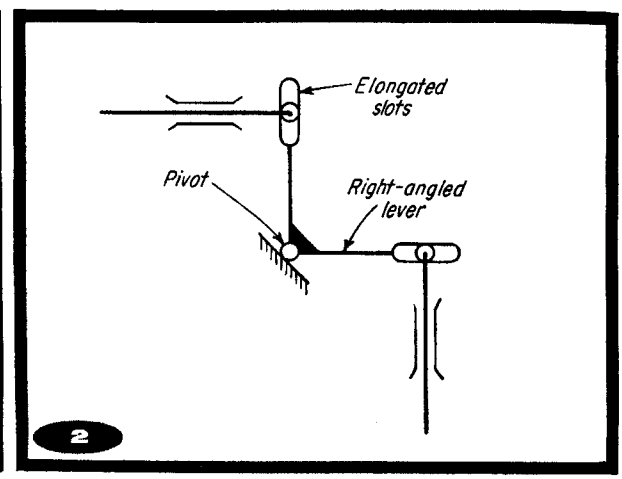
Arrangements of linkages, slides, friction drives and gears that can be the basic of many ingenious devices.

Federico Strasser

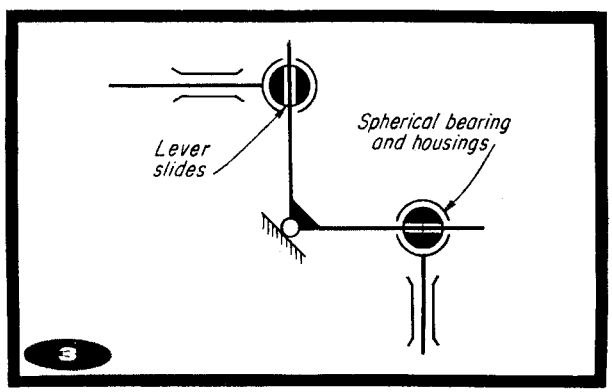
Linkages



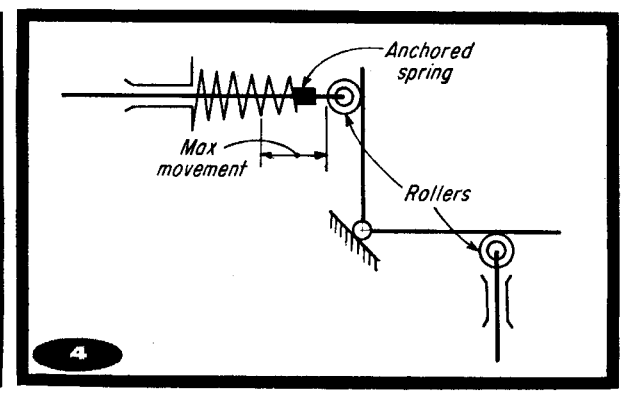
Basic problem (θ is generally close to 90°)



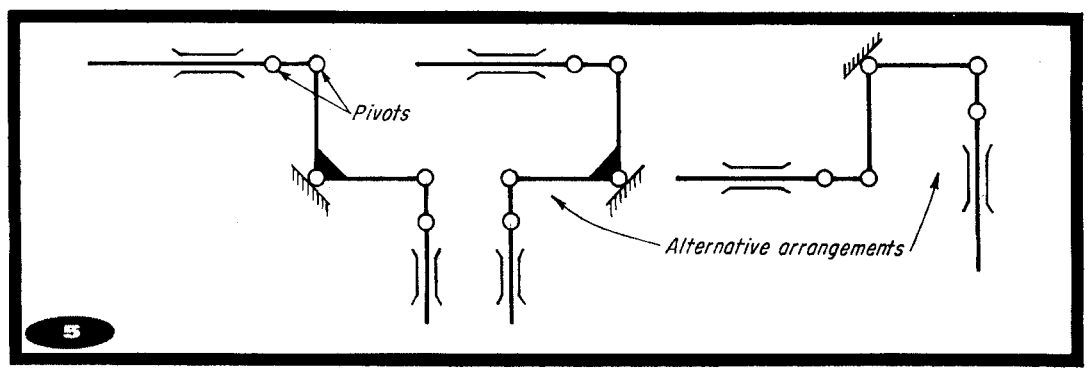
Slotted lever



Spherical bearings

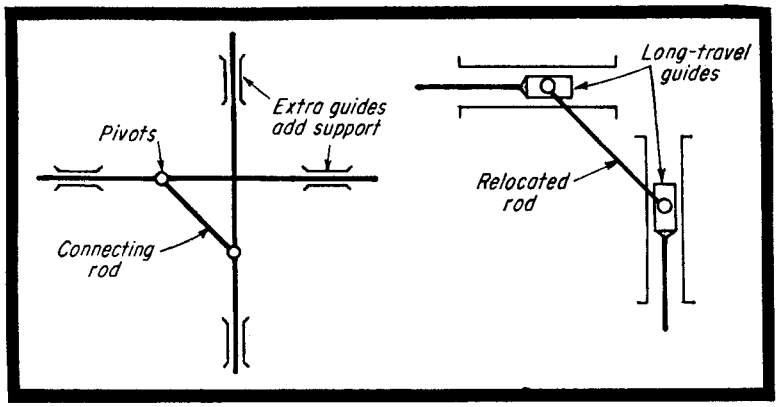


Spring-loaded lever



Pivoted levers with alternative arrangements

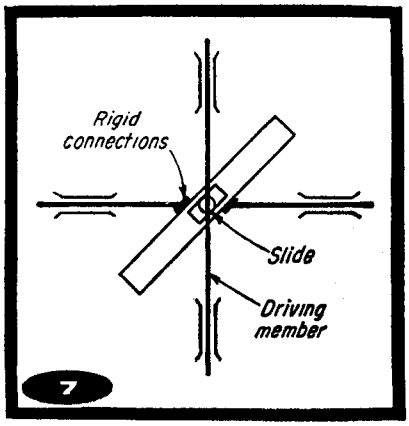
Guides



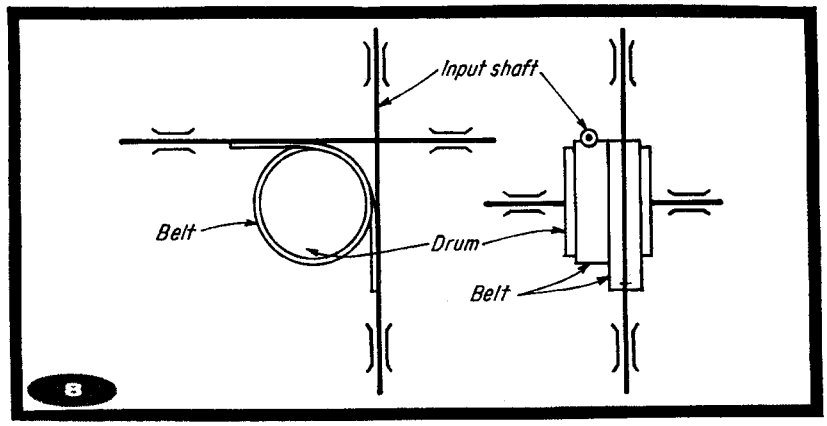
6

Single connecting rod (left) is relocated (right) to get around need for extra guides

Friction Drives

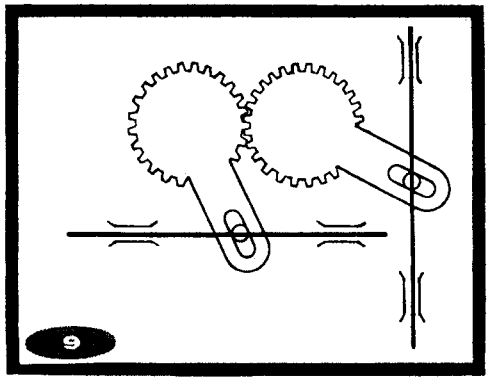


Inclined bearing-guide

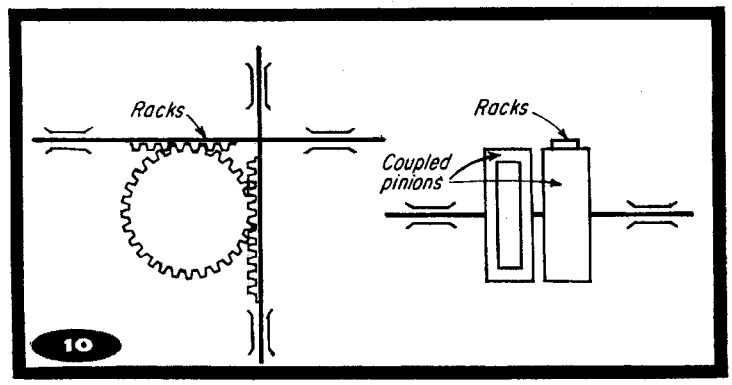


Belt, steel band, or rope around drum, fastened to driving and driven members; sprocket-wheels and chain can replace drum and belt

Gears



Matching gear-segments

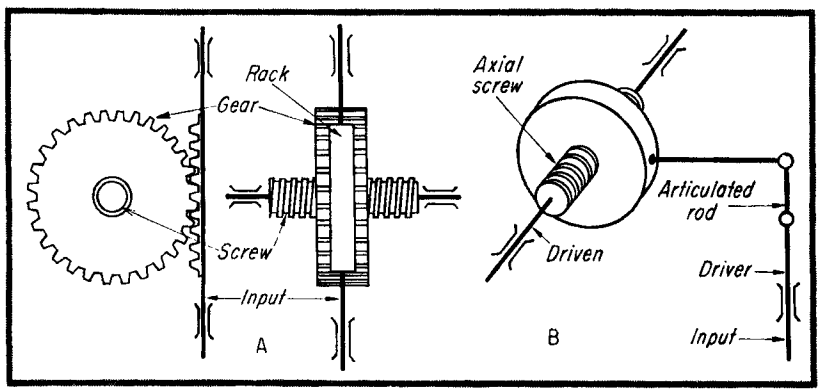


Racks and coupled pinions (can be substituted by friction surfaces for low-cost setup)

9 More Ways to Change Straight-Line Direction

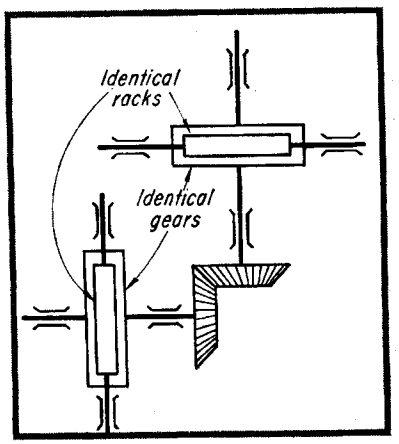
These devices, using gears, cams, pistons, and solenoids, supplement similar arrangements employing linkages, slides, friction drives, and gears, shown.

Federico Strasser



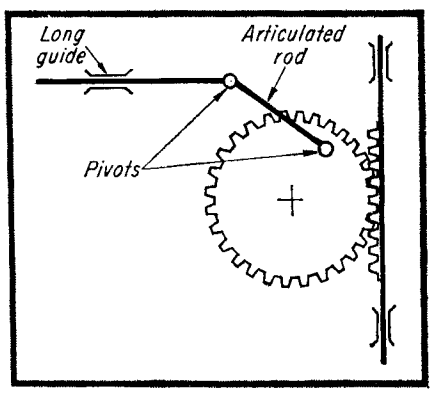
1

Axial screw with rack-actuated gear (A) and articulated driving rod (B) are both irreversible movements, i.e. driver must always drive.



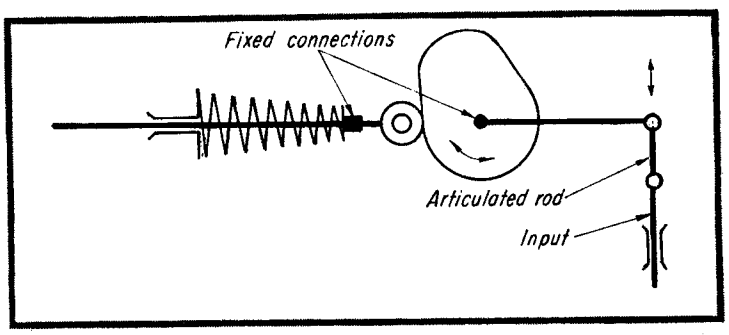
2

Rack-actuated gear with associated bevel gears is reversible.



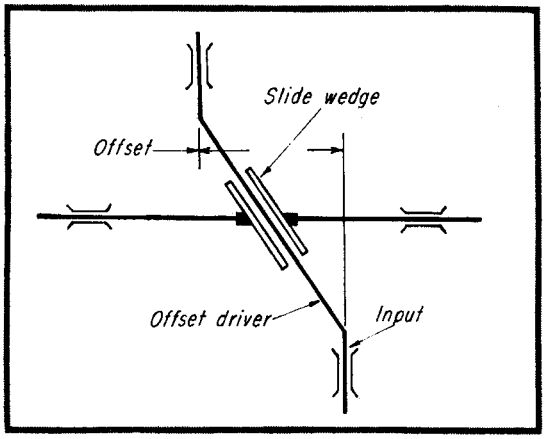
3

Articulated rod on crank-type gear with rack driver. Action is restricted to comparatively short movements.



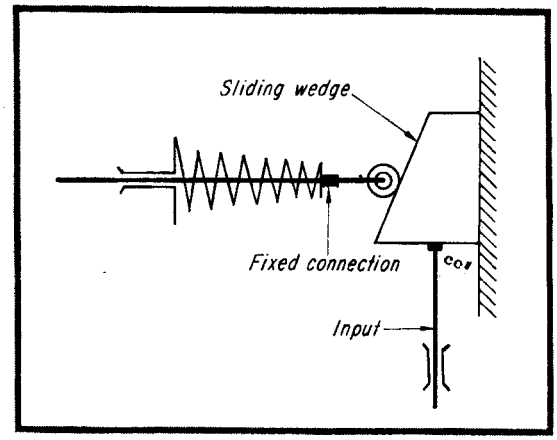
4

Cam and spring-loaded follower allow input/output ratio to be varied according to cam rise. Movement is usually irreversible.



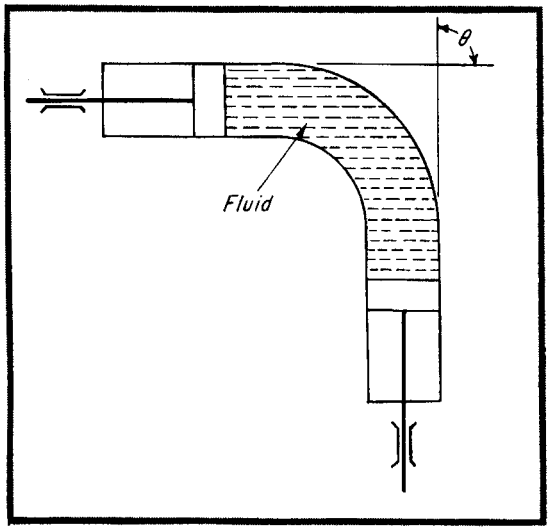
5

Offset driver actuates driven member by wedge action. Lubrication and low coefficient of friction help to allow max offset.



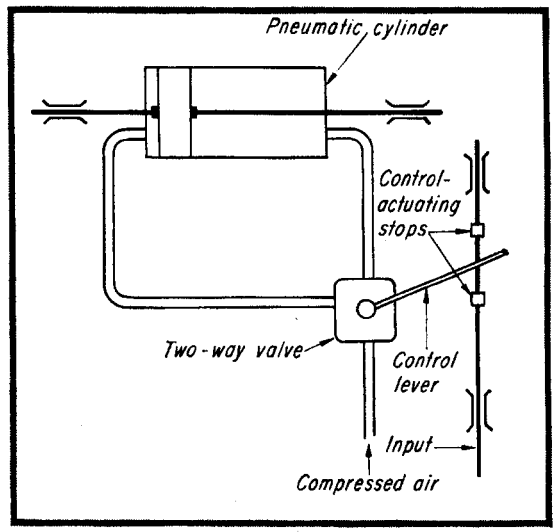
6

Sliding wedge is similar to previous example but requires spring-loaded follower; also, low friction is less essential with roller follower.



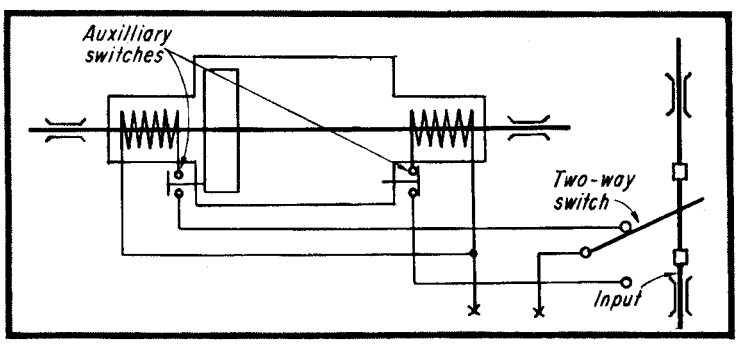
7

Fluid coupling is simple, allows motion to be transmitted through any angle. Leak problems and accurate piston-fitting can make method more expensive than it appears to be. Also, although action is reversible it must always be a compressive one for best results.



8

Pneumatic system with two-way valve is ideal when only two extreme positions are required. Action is irreversible. Speed of driven member can be adjusted by controlling input of air to cylinder.



9

Solenoids and two-way switch are here arranged in analogous device to previous example. Contact to energized solenoid is broken at end of stroke. Again, action is irreversible.

Shape of Pin-Connected Linkages

For symmetrical arrangements with 4 or 5 links.

Philip O. Potts & R. T. Liddicoat

Freely hanging conveyor-parts, special-purpose suspensions and hinged linkages are examples of some mechanical arrangements that can be analyzed quickly by the chart on the continuing page. It has been designed specifically to find the equilibrium shape of 4- or 5-member assemblies (sketched below). When their weights, link lengths, and spans are changed, their angles α and β will change too.

The chart finds these angles for various combinations of weights, spans, or link lengths. It plots the equations:

$$n = 2 \sin \alpha + 2 \sin \beta$$
$$m = 1 + (2 \sin \alpha + 2 \sin \beta)$$
$$\frac{\tan \alpha}{\tan \beta} = \frac{2W_2}{W_1} + 1$$
$$\frac{\tan \alpha}{\tan \beta} = \frac{Q_2}{Q_1} + 1$$

These are for a practical range of n and m, W and Q, up to 90° and 70° for α and β respectively. The equations are derived from the geometry of the mechanism and a "virtual work" principle, and where the tangent curves and sine curves intersect are values for α and β.

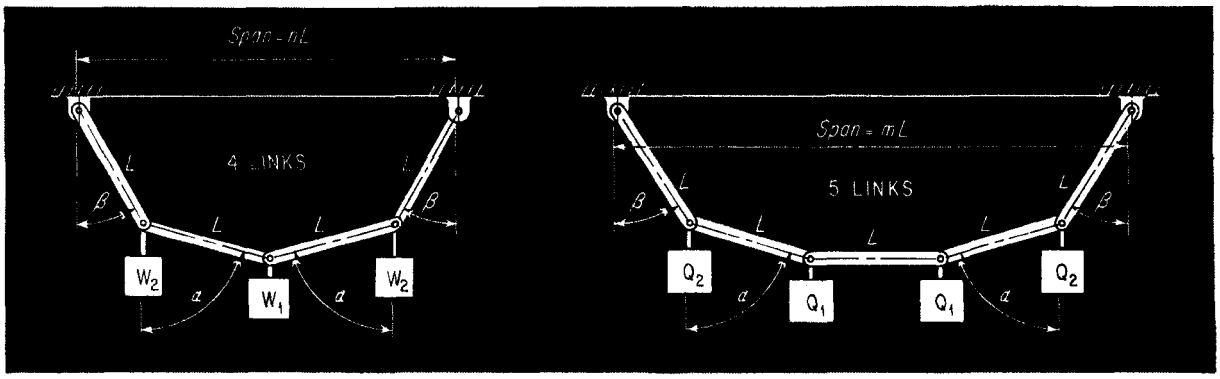
Since equations for the two linkages are similar, any tangent curve holds for a value of Q_2/Q_1 that is twice W_2/W_1, and any sine curve holds for a value of n that is one less than m.

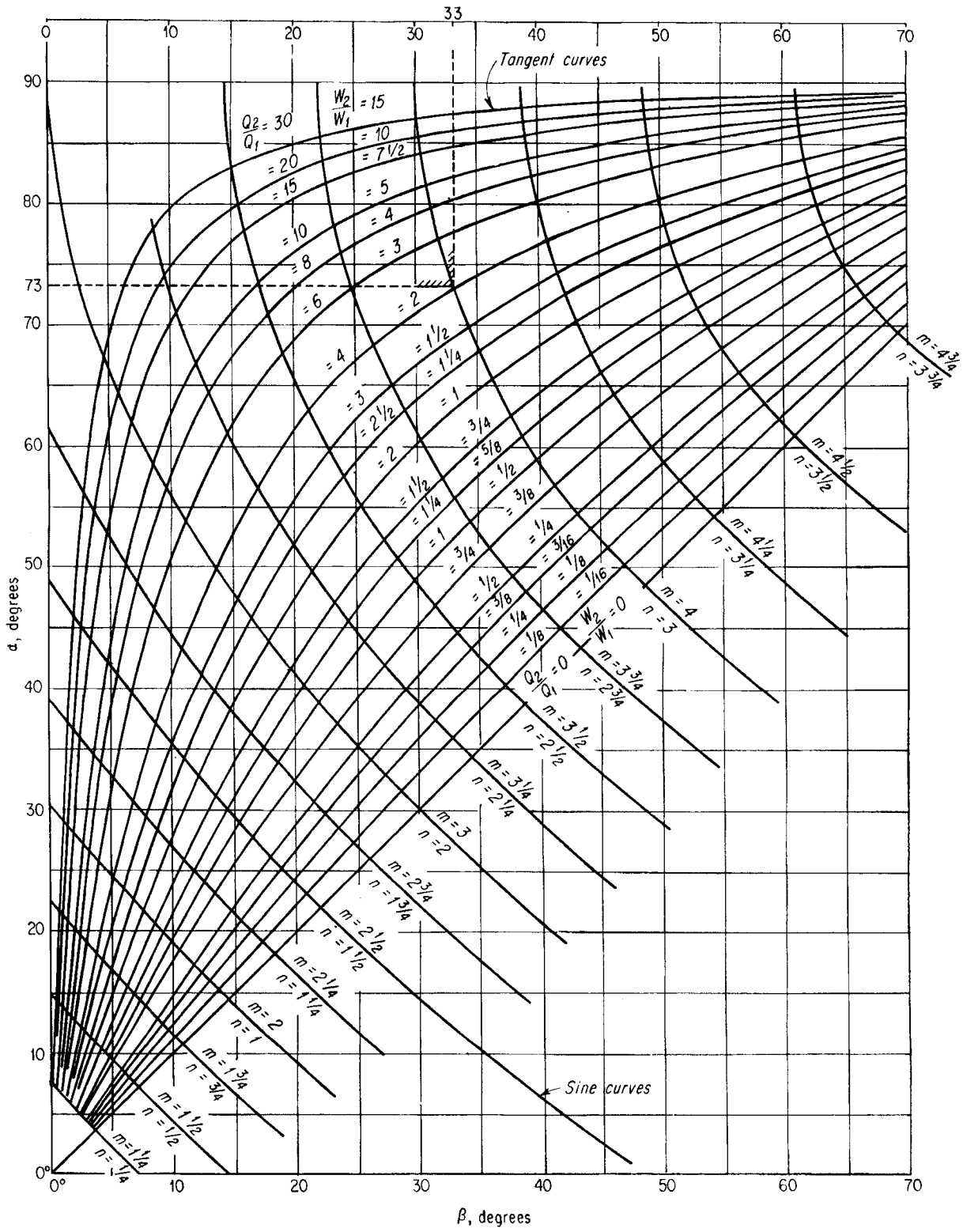
Example: In a 4-link system, the span is $3L$; W_2 loads are twice W_1. Follow the sine curve for $n = 3$, and the tangent curve for $W_2/W_1 = 2$. They intersect at $\alpha = 73°$, $\beta = 33°$. If the system had 5 links, the same angles would be read for a span length of $4L$ and a load ratio of $Q_2/Q_1 = 4$.

Results can also be determined in reverse—if a certain shape or position of a linkage is needed, the necessary loading or span to give required angles can be read directly from the chart. Also, the linkages may be oriented in any direction as long as links are in tension and loads are perpendicular to span. Forces produced by other means than gravity may be substituted for weights.

Link weight may either be neglected or assumed added into W or Q: n is any positive quantity between 0 and 4; m is any positive quantity between 1 and 5.

Equilibrium shape . . .
of hanging, pin-connected linkages depends on weight ratio and link-lengths. Symmetrical weights, and equi-length links in relation to span length are assumed in analytic method described here.




33
Tangent curves
Sine curves
α, degrees
β, degrees
W2/W1 = 15
10
7 1/2
5
4
3
2
1 1/2
1 1/4
1
3/4
5/8
1/2
3/8
1/4
3/16
1/8
1/16
0
Q2/Q1 = 30
20
15
10
8
6
4
3
2 1/2
2
1 1/2
1 1/4
1
3/4
1/2
5/8
1/4
1/8
m = 4 3/4
n = 3 3/4
m = 4 1/2
n = 3 1/2
m = 4 1/4
n = 3 1/4
m = 4
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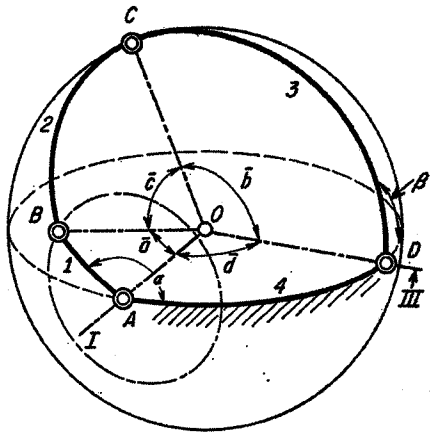
n = 3/4
m = 1 1/2
n = 1/2
m = 1 1/4
n = 1/4

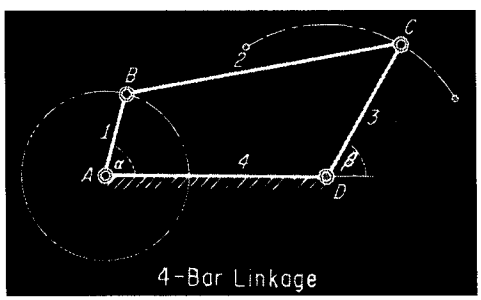
Seven Popular Types of Three-Dimensional Drives

Main advantage of three-dimensional drives is their ability to transit motion between nonparallel shafts. They can also generate other types of helpful motion. With this roundup are descriptions of industrial applications.

Dr. W. Meyer Zur Capellen



The Spherical Crank



1 spherical crank drive

This type of drive is the basis for most 3-D linkages, much as the common 4-bar linkage is the basis for the two-dimensional field. Both mechanisms operate on similar principles. (In the accompanying sketches, α is the input angle, and β the output angle. This notation has been used throughout the article.)

In the 4-bar linkage, the rotary motion of driving crank *1* is transformed into an oscillating motion of output link 3. If the fixed link is made the shortest of all, then you have a double-crank mechanism, in which both the driving and driven members make full rotations.

In the spherical crank drive, link *1* is the input, link 3 the output. The axes of rotation intersect at point *O*; the lines connecting *AB*, *BC*, *CD* and *DA* can be thought of as part of great circles of a sphere. The length of the link is best represented by angles *a*, *b*, *c* and *d*.

2 spherical-slide oscillator

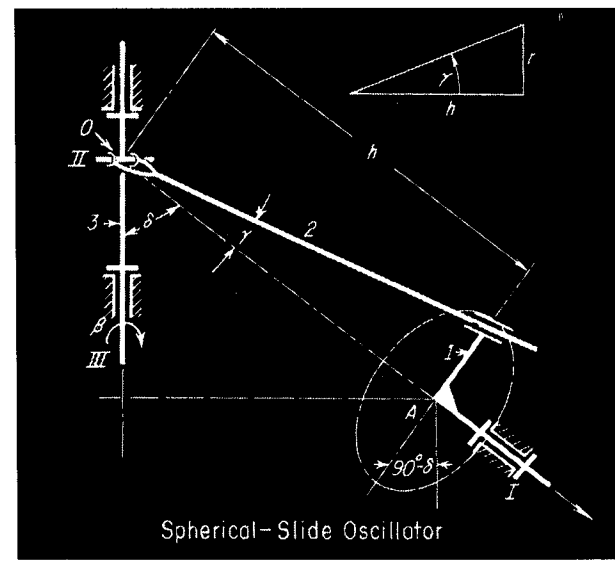
The two-dimensional slider crank is obtained from a 4-bar linkage by making the oscillating arm infinitely long. By making an analogous change in the spherical crank, you can obtain the spherical slider crank shown at right.

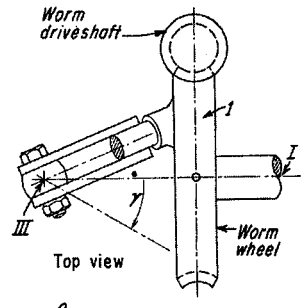
The uniform rotation of input shaft I is transferred into a nonuniform oscillating or rotating motion of output shaft III. These shafts intersect at an angle δ corresponding to the frame link 4 of the spherical crank. Angle γ corresponds to length of link *1*. Axis II is at right angle to axis III.

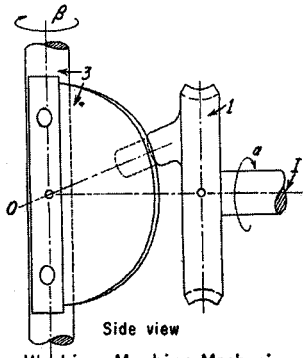
The output oscillates when γ is smaller than δ; the output rotates when γ is larger than δ.

Relation between input angle α and output angle β is (as designated in skewed Hook's joint, below)

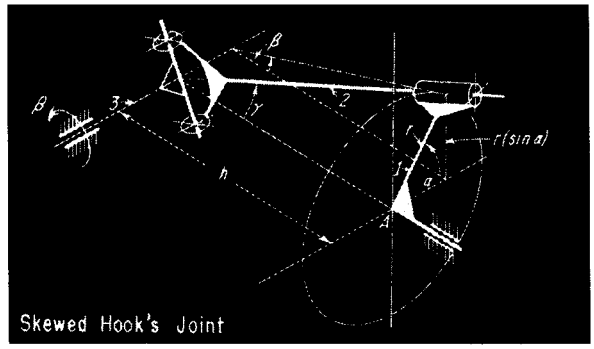
$$\tan \beta = \frac{(\tan \gamma)(\sin \alpha)}{\sin \delta + (\tan \gamma)(\cos \delta)(\cos \alpha)}$$

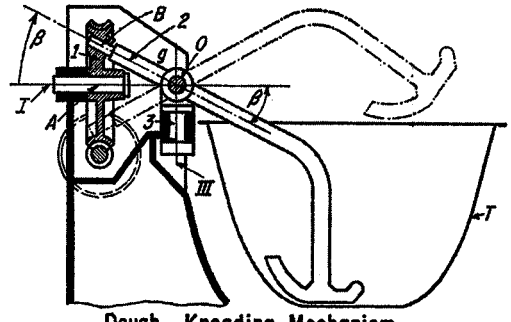






Washing-Machine Mechanism





Dough-Kneading Mechanism

3 skewed hook's joint

This variation of the spherical crank is often used where an almost linear relation is desired between input and output angles for a large part of the motion cycle.

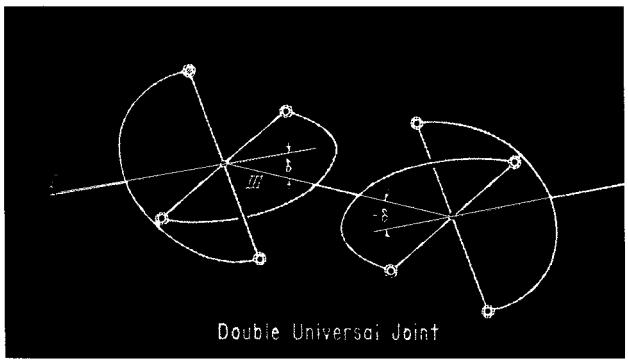
The equation defining the output in terms of the input can be obtained from the above equation by making $\delta = 90°$. Thus $\sin \delta = 1$, $\cos \delta = 0$, and

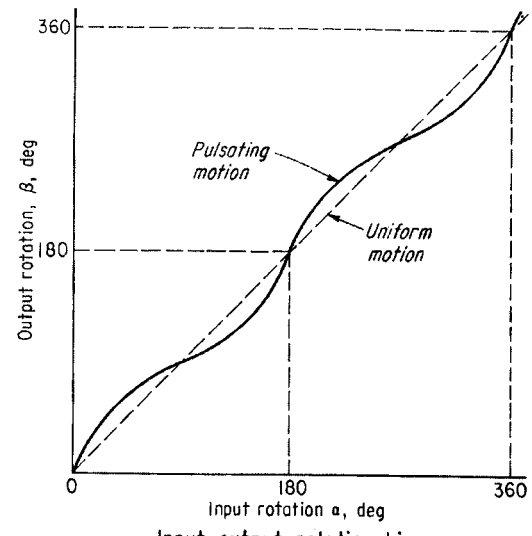
$$\tan \beta = \tan \gamma \sin \alpha$$

The principle of the skewed Hook's joint has been recently applied to the drive of a washing machine (see sketch at left).

Here, the driveshaft drives the worm wheel 1 which has a crank fashioned at an angle γ. The crank rides between two plates and causes the output shaft III to oscillate in accordance with the equation above.

The dough-kneading mechanism at right is also based on the Hook's joint, but utilizes the path of link 2 to give a wobbling motion that kneads dough in the tank.





Input-output relationship

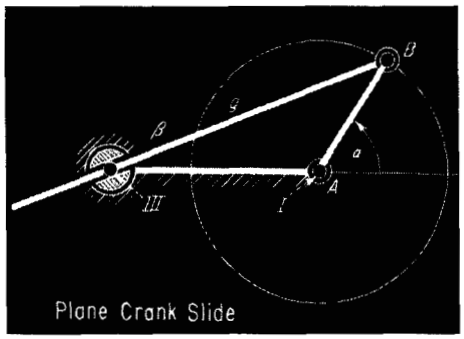
4 the universal joint

The universal joint is a variation of the spherical-slide oscillator, but with angle $\gamma = 90°$. This drive provides a totally rotating output and can be operated as a pair, as shown above.

Equation relating input with output for a single universal joint, where δ is angle between connecting link and shaft I:

$$\tan \beta = \tan \alpha \cos \delta$$

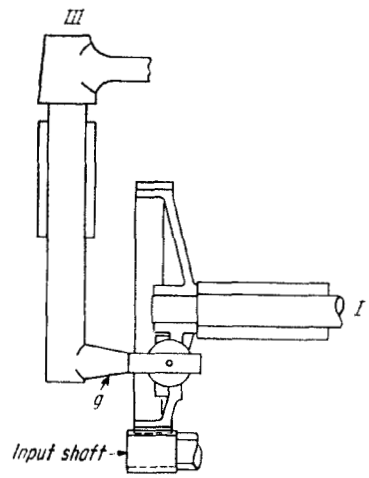
Output motion is pulsating (see curve) unless the joints are operated as pairs to provide a uniform motion.



5 the 3-D crank slide

The three-dimensional crank slide is a variation of a plane crank slide (see sketch), with a ball point through which link g always slides, while a point B on link g describes a circle. A 3-D crank is obtained from this mechanism by making output shaft III not normal to the plane of the circle; another way is to make shafts I and III nonparallel.

A practical variation of the 3-D crank slide is the agitator mechanism (right). As input gear I rotates, link g swivels around (and also lifts) shaft III. Hence, vertical link has both an oscillating rotary motion and a sinusoidal harmonic translation in the direction of its axis of rotation. The link performs what is essentially a screw motion in each cycle.



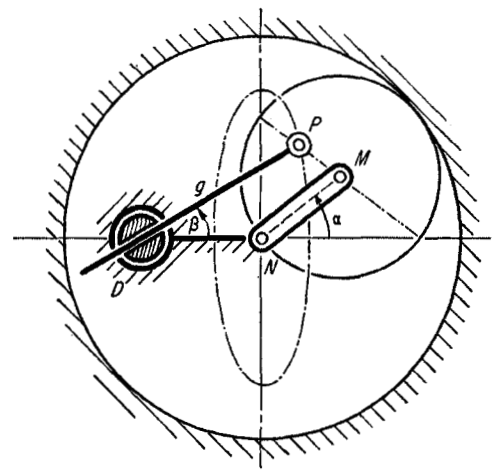
Agitator Mechanism

6 the elliptical slide

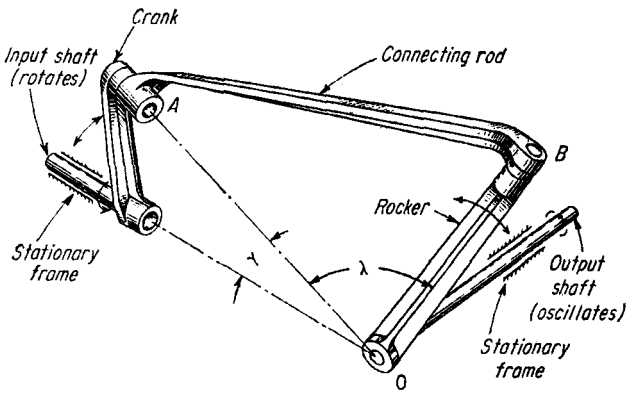
The output motion, β, of a spherical slide oscillator, p23-12, can be duplicated by means of a two-dimensional "elliptical slide." The mechanism has a link g which slides through a pivot point D and is fastened to a point P moving along an elliptical path. The ellipse can be generated by a Cardan drive, which is a planetary gear system with the planet gear half the diameter of the internal gear. The center of the planet, point M, describes a circle; any point on its periphery describes a straight line, and any point in between, such as point P, describes an ellipse.

There are certain relationships between the dimensions of the 3-D spherical slide and the 2-D elliptical slide: $\tan \gamma / \sin \delta = a/d$ and $\tan \gamma / \cot \delta = b/d$, where a is the major half-axis, b the minor half-axis of the ellipse, and d is the length of the fixed link DN. The minor axis lies along this link.

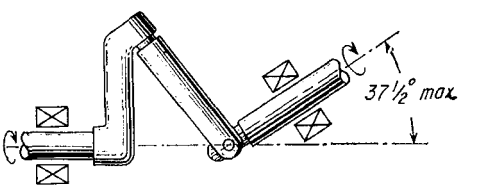
If point D is moved within the ellipse, a completely rotating output is obtained, corresponding to the rotating spherical crank slide.



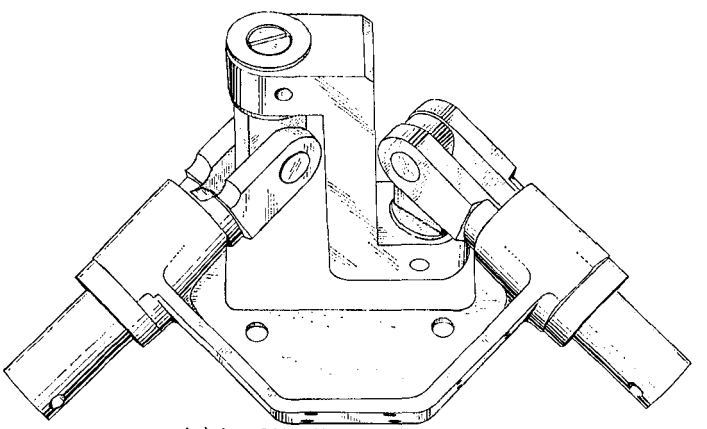
The Elliptical Slide



(A) Basic Configuration



(B) Its Inversion



(C) As a 90° Uniform Motion Transmitter

7 the space crank

One of the most recent developments in 3-D linkages is the space crank shown in (A) see also PE—"Introducing the Space Crank—a New 3-D Mechanism," Mar 2 '59. It resembles the spherical crank discussed on page 76, but has different output characteristics. Relationship between input and output displacements is:

$$\cos \beta = (\tan \gamma)(\cos \alpha)(\sin \beta) - \frac{\cos \lambda}{\cos \gamma}$$

Velocity ratio is:

$$\frac{\omega_o}{\omega_i} = \frac{\tan \gamma \sin \alpha}{1 + \tan \gamma \cos \alpha \cot \beta}$$

where ω_o is the output velocity and ω_i is the constant input velocity.

An inversion of the space crank is shown in (B). It can couple intersecting shafts, and permits either shaft to be driven with full rotations. Motion is transmitted up to 37½° misalignment.

By combining two inversions, (C), a method for transmitting an exact motion pattern around a 90° bend is obtained. This unit can also act as a coupler or, if the center link is replaced by a gear, it can drive two output shafts; in addition, it can be used to transmit uniform motion around two bends.

Transmission Linkages for Multiplying Short Motions

THE ACCOMPANYING SKETCHES show typical mechanisms for multiplying short linear motions, usually converting the linear motion into rotation. Although the particular mechanisms shown are designed to multiply the movements of diaphragms or bellows, the same or similar constructions have possible applications wherever it is required to obtain greatly multiplied motions. These patented transmissions depend on cams, sector gears and pinions, levers and cranks, cord or chain, spiral or screw feed, magnetic attraction, or combinations of these devices.

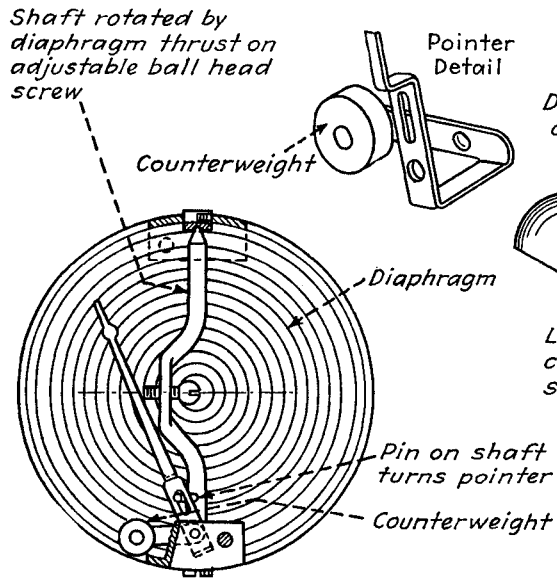


FIG. 1 – *Lever type transmission in pressure gage*

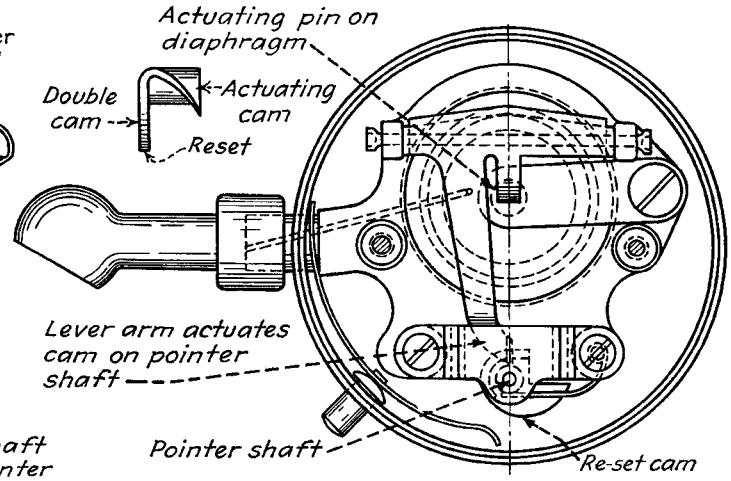


FIG. 2 – *Lever and cam drive for tire gage*

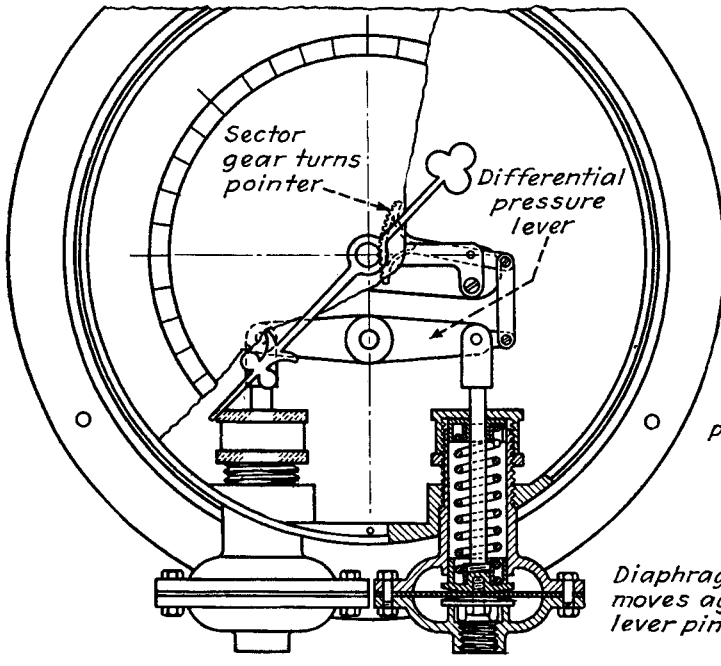


FIG. 3 – *Lever and sector gear in differential pressure gage*

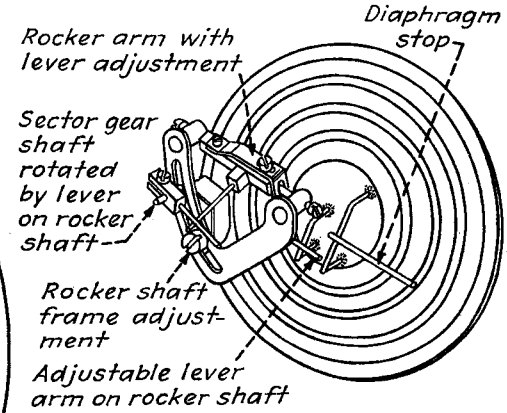

Diaphragm stop
Rocker arm with lever adjustment
Sector gear shaft rotated by lever on rocker shaft
Rocker shaft frame adjustment
Adjustable lever arm on rocker shaft

FIG. 4 – *Sector gear drive for aircraft airspeed indicator*

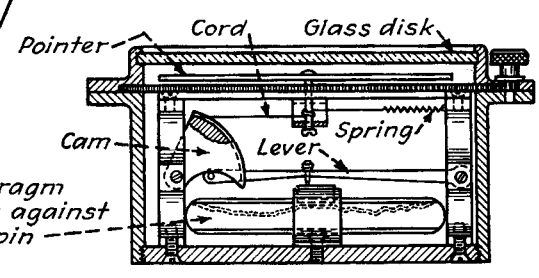


FIG. 5 – *Lever, cam and cord transmission in barometer*

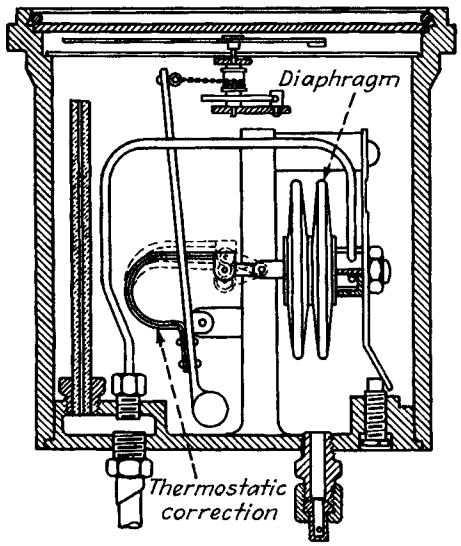


FIG. 6 – *Link and chain transmission for rate of climb instrument*

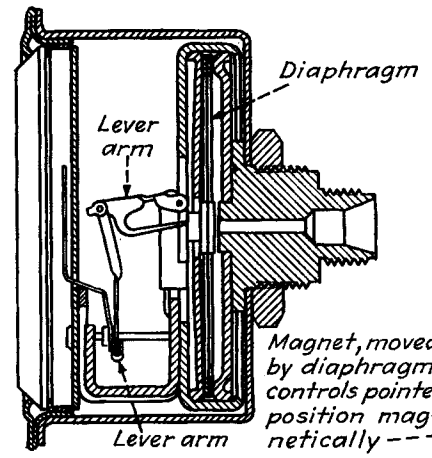


FIG. 7 – *Lever system in automobile gasoline tank gage*

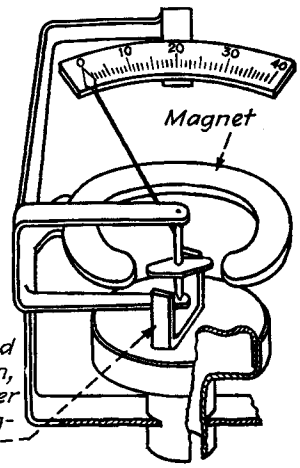


FIG. 8 – *Interfering magnetic fields for fluid pressure measurement*

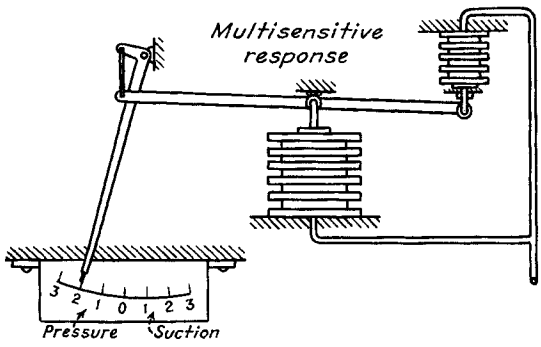


FIG. 9 – *Lever system for atmospheric pressure variations*

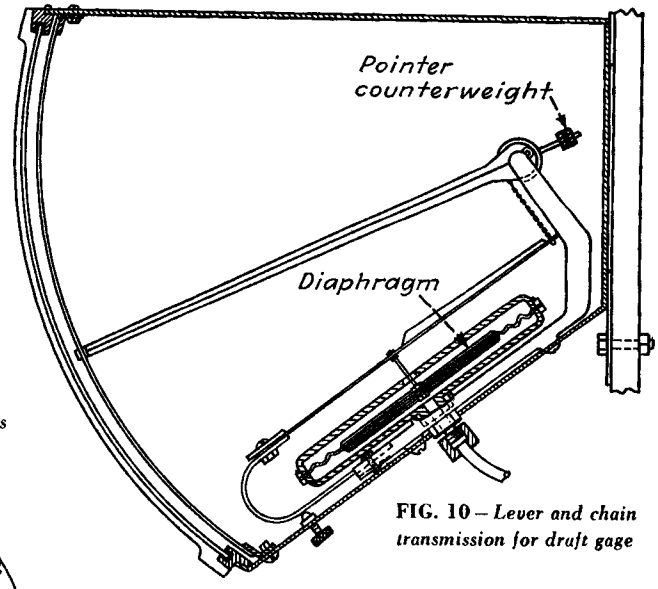


FIG. 10 – *Lever and chain transmission for draft gage*

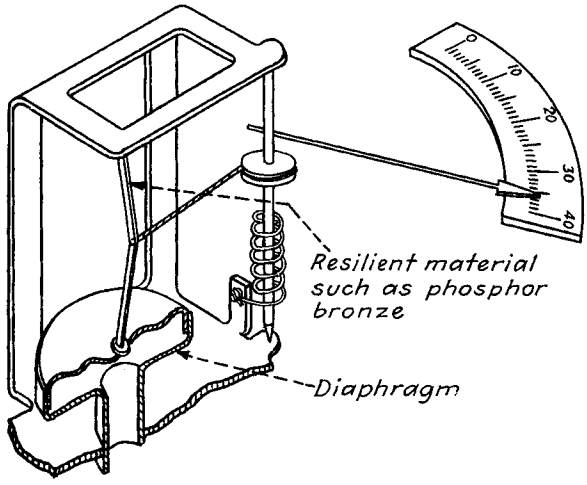


FIG. 11 – *Toggle and cord drive for fluid pressure instrument*

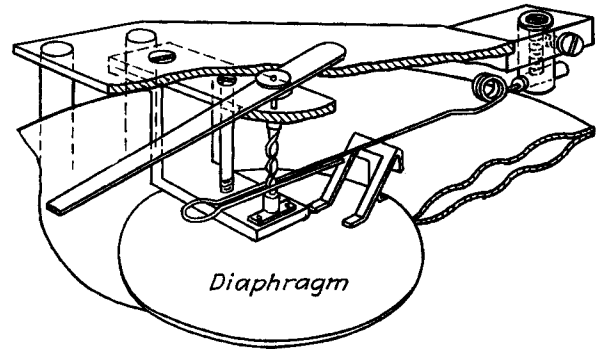


FIG. 12 – *Spiral feed transmission for general purpose instrument*

Power Thrust Linkages and Their Applications

POWERED STRAIGHT LINE MOTION over short distances is applicable to many types of machines or devices for performing specialized services. These motions can be produced by a steam, pneumatic or hydraulic cylinder, or by a self-contained electric powered unit such as the General Electric Thrustors shown herewith. These Thrustors may be actuated manually by pushbuttons, or automatically by mechanical devices or the photo-electric relay as with a door opener. These illustrations will suggest many other arrangements.

Fig. 1—Transfer motion to distant point.

Fig. 2—Double throw by momentary applications.

Fig. 3—Trammel plate divides effort and changes directions of motion.

Fig. 4—Constant thrust toggle. Pressure is same at all points of throw.

Fig. 5—Multiplying motion, 6 to 1, might be used for screen shift.

Fig. 6—Bell crank and toggle may be applied in embossing press, extruder or die-caster.

Fig. 7—Horizontal pull used for clay pigeon traps, hopper trips, and sliding elements with spring or counterweighted return.

Fig. 8—Shipper rod for multiple and distant operation as series of valves.

Fig. 9—Door opener. Upthrust of helical racks rotates gear and arm.

Fig. 10—Accelerated motion by shape of cam such as on a forging hammer.

Fig. 11—Intermittent lift as applied to lifting pipe from well.

Fig. 12—Straight-line motion multiplied by pinion and racks.

Fig. 13—Rotary motion with cylindrical cam. Operates gate on conveyor belt.

Fig. 14—Thrust motions and "dwells" regulated by cam.

Fig. 15—Four positive positions with two Thrustors.

Fig. 16—Toggle increasing thrust at right angle.

Fig. 17—Horizontal straight-line motion as applied to a door opener.

Fig. 18—Thrusts in three directions with two Thrustors.

Fig. 19—Fast rotary motion using step screw and nut.

Fig. 20—Intermittent rotary motion. Operated by successive pushing of operating button, either manually or automatically.

Fig. 21—Powerful rotary motion with worm driven by rack and pinion.

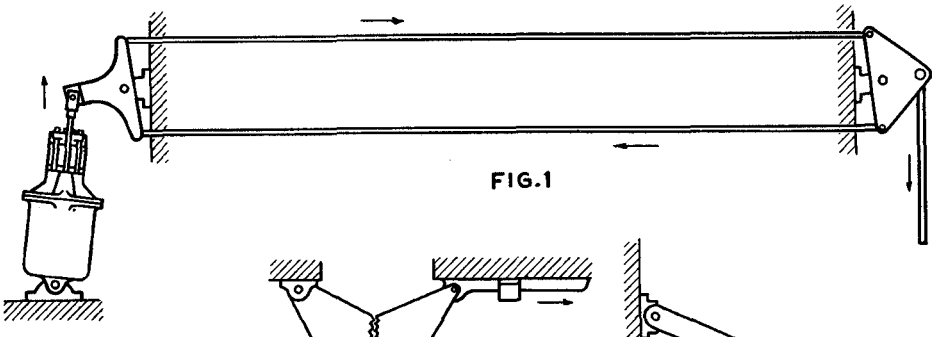


FIG.1

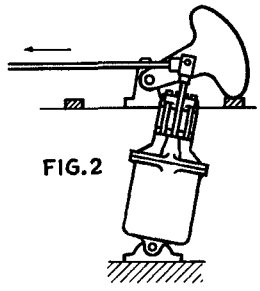


FIG.2

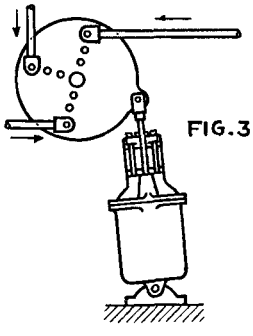


FIG.3

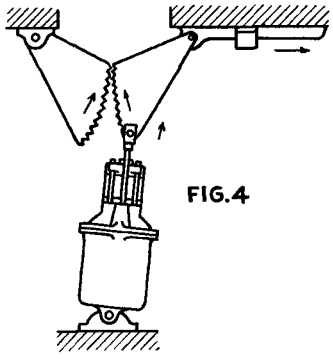


FIG.4

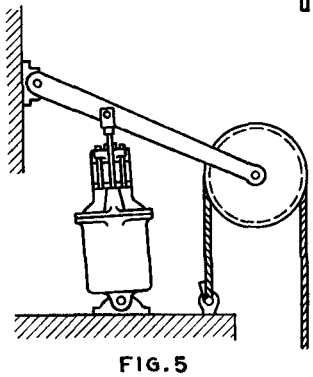
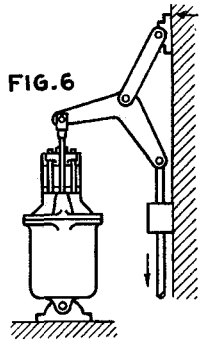


FIG.5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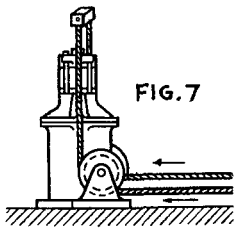


FIG.7

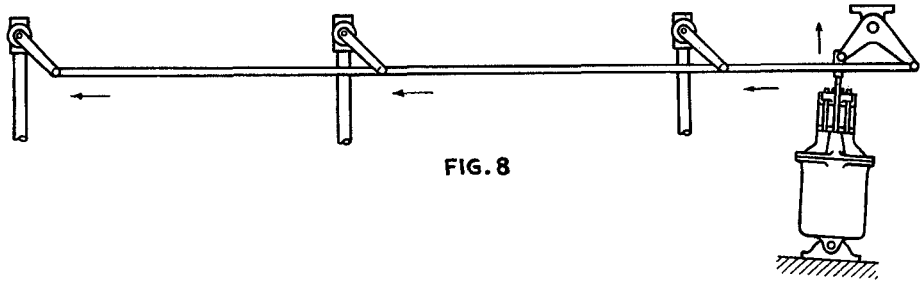


FIG.8

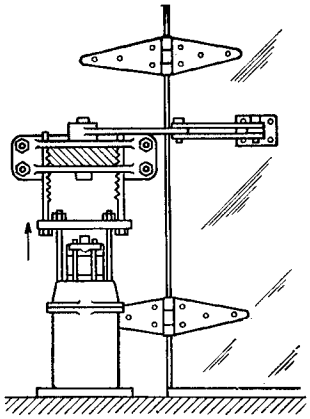


FIG. 9

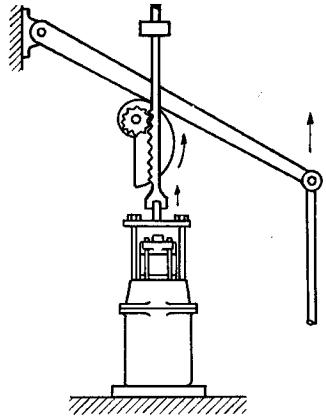


FIG. 10

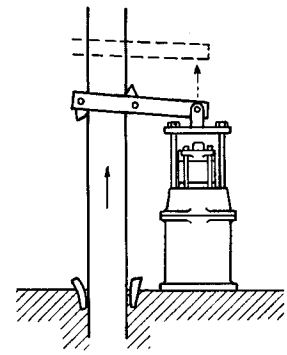


FIG. 11

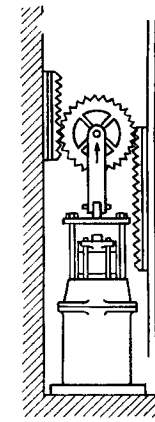


FIG. 12

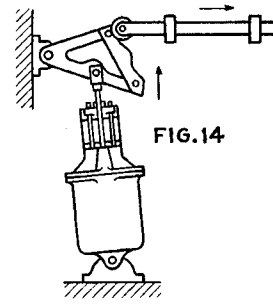


FIG. 14

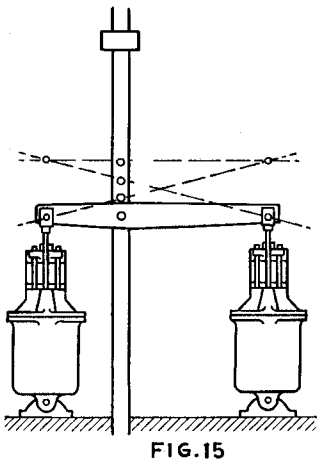


FIG. 15

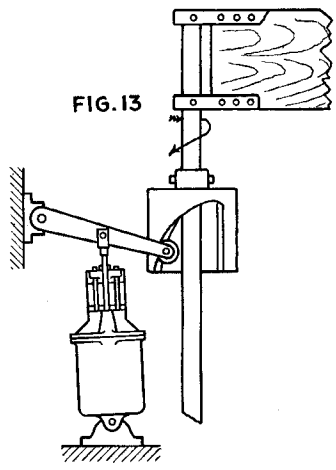


FIG. 13

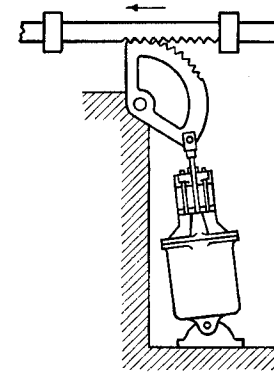


FIG. 17

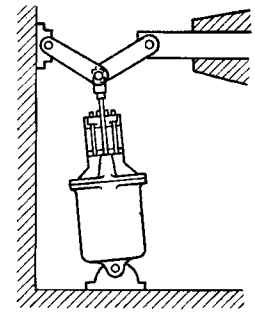


FIG. 16

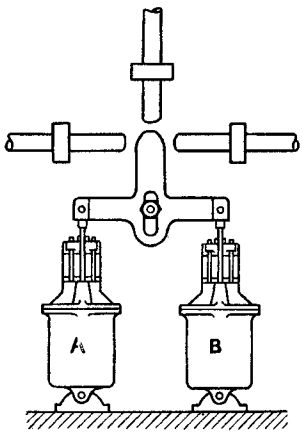


FIG. 18

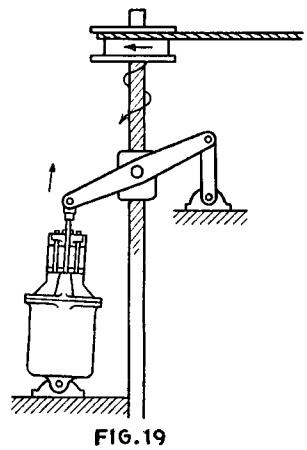


FIG. 19

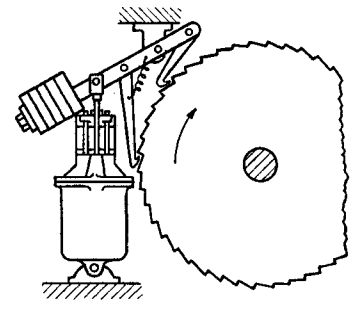


FIG. 20

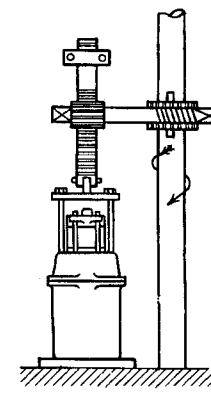


FIG. 21

Toggle Linkage Applications in Different Mechanisms

Thomas P. Goodman

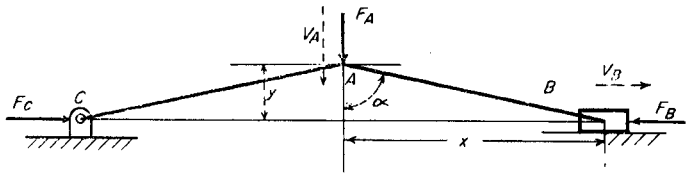


Fig. 1 Mechanical advantage = $\frac{F_B}{F_A} = \frac{l}{2}\frac{x}{y} = \frac{l}{2}\,Tan\,\alpha = \frac{V_A}{V_B}$

MANY MECHANICAL LINKAGES are based on the simple toggle which consists of two links that tend to line-up in a straight line at one point in their motion. The mechanical advantage is the velocity ratio of the input point A to the output point B: or V_A/V_B. As the angle α approaches 90 deg, the links come into toggle and the mechanical advantage and velocity ratio both approach infinity. However, frictional effects reduce the forces to much less than infinity although still quite high.

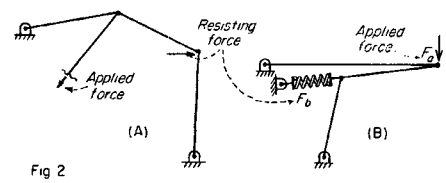


Fig 2

FORCES CAN BE APPLIED through other links, and need not be perpendicular to each other. (A) One toggle link can be attached to another link rather than to a fixed point or slider. (B) Two toggle links can come into toggle by lining up on top of each other rather than as an extension of each other. Resisting force can be a spring force.

HIGH MECHANICAL ADVANTAGE

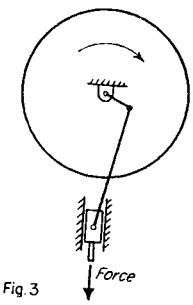
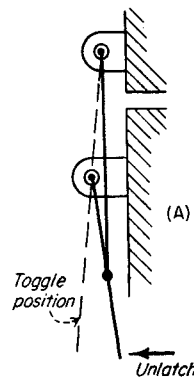


Fig. 3

IN PUNCH PRESSES, large forces are needed at the lower end of the work-stroke, however little force is required during the remainder. Crank and connecting rod come into toggle at the lower end of the punch stroke, giving a high mechanical advantage at exactly the time it is most needed.



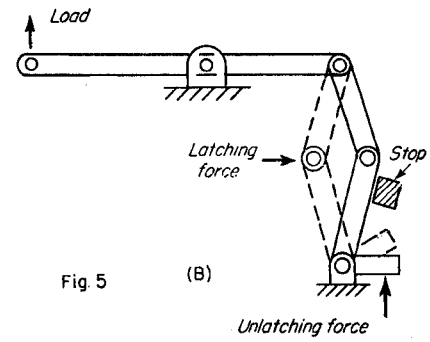
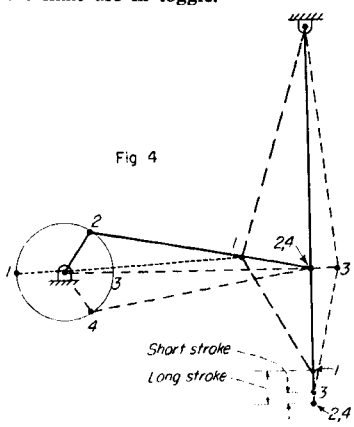
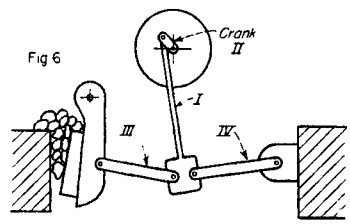


Fig. 5

LOCKING LATCHES produce a high mechanical advantage when in the toggle portion of the stroke. (A) Simple latch exerts a large force in the locked position. (B) For positive locking, closed position of latch is slightly beyond toggle position. Small unlatching force opens linkage.

COLD-HEADING RIVET MACHINE is designed to give each rivet two successive blows. Following the first blow (point 2) the hammer moves upward a short distance (to point 3), to provide clearance for moving the workpiece. Following the second blow (at point 4), the hammer then moves upward a longer distance (to point 1). Both strokes are produced by one revolution of the crank and at the lowest point of each stroke (points 2 and 4) the links are in toggle.





STONE CRUSHER uses two toggle linkages in series to obtain a high mechanical advantage. When the vertical link I reaches the top of its stroke, it comes into toggle with the driving crank II; at the same time, link III comes into toggle with link IV. This multiplication results in a very large crushing force.

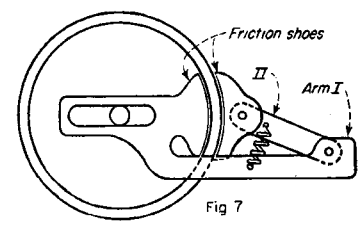


Fig 7

FRICTION RATCHET is mounted on a wheel; light spring keeps friction shoes in contact with the flange. This device permits clockwise motion of the arm I. However, reverse rotation causes friction to force link II into toggle with the shoes which greatly increases the locking pressure.

HIGH VELOCITY RATIO

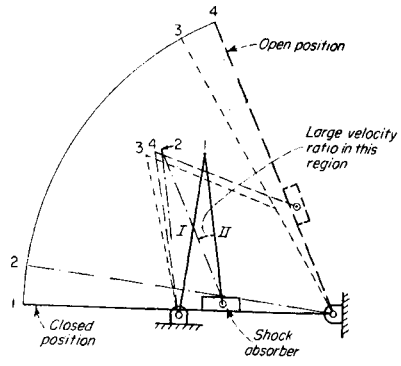


Fig. 8

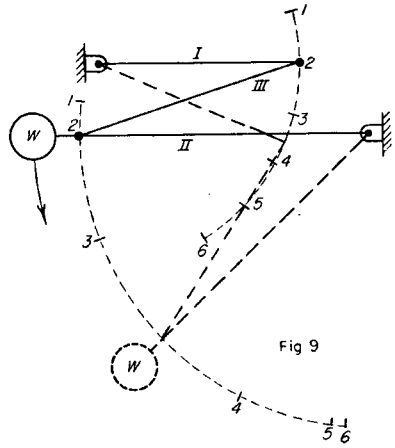


Fig 9

DOOR CHECK LINKAGE gives a high velocity ratio at one point in the stroke. As the door swings closed, connecting link I comes into toggle with the shock absorber arm II, giving it a large angular velocity. Thus, the shock absorber is more effective retarding motion near the closed position.

IMPACT REDUCER used on some large circuit breakers. Crank I rotates at constant velocity while lower crank moves slowly at the beginning and end of the stroke. It moves rapidly at the mid stroke when arm II and link III are in toggle. Falling weight absorbs energy and returns it to the system when it slows down.

VARIABLE MECHANICAL ADVANTAGE

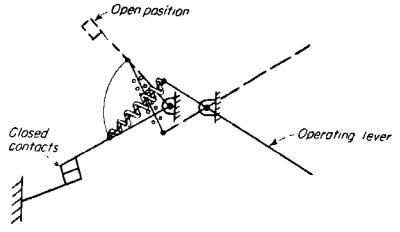


Fig 10

TOASTER SWITCH uses an increasing mechanical advantage to aid in compressing a spring. In the closed position, spring holds contacts closed and the operating lever in the down position. As the lever is moved upward, the spring is compressed and comes into toggle with both the contact arm and the lever. Little effort is required to move the links through the toggle position; beyond this point, the spring snaps the contacts closed.

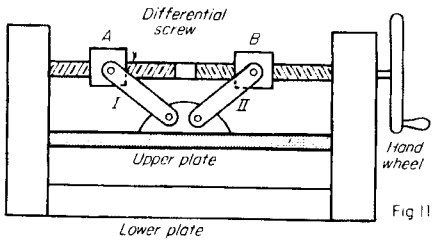
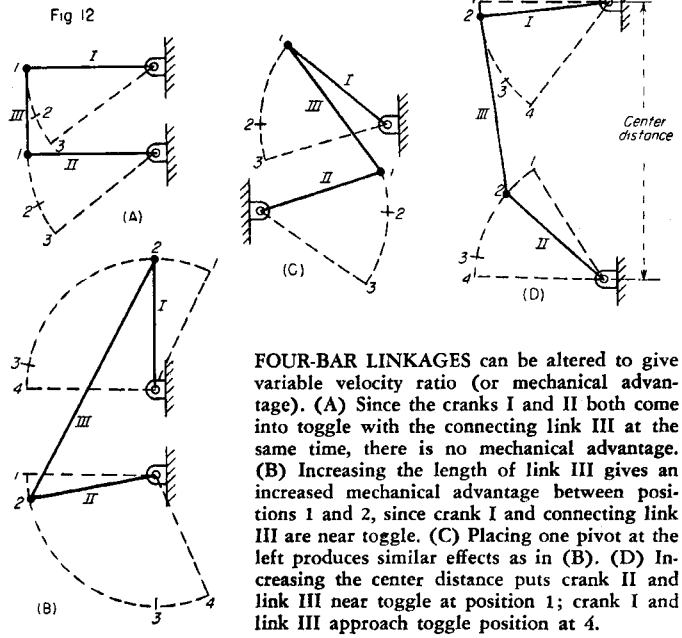


Fig 11

TOGGLE PRESS has an increasing mechanical advantage to counteract the resistance of the material being compressed. Rotating handwheel with differential screw moves nuts *A* and *B* together and links I and II are brought into toggle.

Fig 12



FOUR-BAR LINKAGES can be altered to give variable velocity ratio (or mechanical advantage). (A) Since the cranks I and II both come into toggle with the connecting link III at the same time, there is no mechanical advantage. (B) Increasing the length of link III gives an increased mechanical advantage between positions 1 and 2, since crank I and connecting link III are near toggle. (C) Placing one pivot at the left produces similar effects as in (B). (D) Increasing the center distance puts crank II and link III near toggle at position 1; crank I and link III approach toggle position at 4.

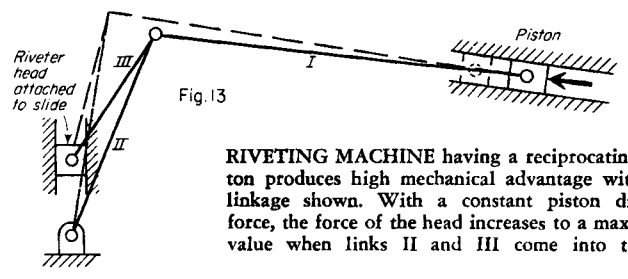


Fig.13

RIVETING MACHINE having a reciprocating piston produces high mechanical advantage with the linkage shown. With a constant piston driving force, the force of the head increases to a maximum value when links II and III come into toggle.

When Linkages Need Harmonic Analysis

Feeding the working equation into a computer provided the handy table that accompanies this article. Now you'll find it easier to perform the harmonic analysis needed for efficient crank-and-rocker mechanisms.

Dr. Ferdinand Freudenstein & K. Mohan

The proportions of linkages can be determined readily enough by graphical methods. But when you have to know how these proportions affect the higher harmonics and inertia forces, complex analytical methods are very useful.

Equations for harmonic analysis of the general crank-and-rocker linkage were developed (by author Freudenstein) in *Journal of Applied Mechanics*, Dec '59, p 673. Those equations established procedures for the calculation of various coefficients and terms needed to determine whether a given mechanism has significant overtones.

But those equations still require considerable computations. Because crank-and-rocker mechanisms are important in the design of high-speed machinery, we carried the work a step further by programing on an IBM 650 computer the following:

Working equation for output angle:

$$\psi = K + \sum_{m=1}^{\infty} \frac{(-\tan\tfrac{1}{2}u)^m}{(-m)} \sin m\phi - \frac{C_0}{4} \sin u \cos\phi -$$

$$\sum_{m=1}^{\infty} \frac{\sin u}{4m} (C_{m-1} - C_{m+1}) \cos m$$

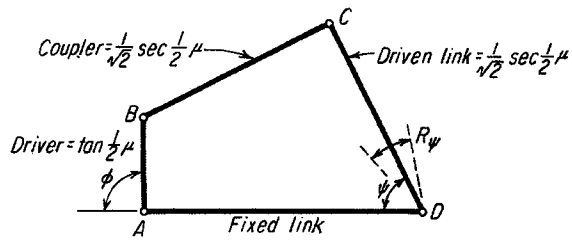
It was with this equation that we obtained the table that accompanies this article.

The rotation of rocker or output link (see diagram) has been expressed as a Fourier Series in terms of the input-crank rotation angle ϕ, and the proportions of the linkage are given in terms of a design parameter u. This parameter is the maximum value of the "pressure angle," which in the diagram is the absolute value of angle BCD

SYMBOLS

A_G = acceleration of gyro
a_m = amplitude of mth sine series term, radians
b_m = amplitude of mth cos series term, radians
c_m = amplitude of mth harmonic of rocker motion, radians
d_m = amplitude ratio, mth to 1st harmonic of rocker motion
f_m = amplitude ratio, mth to 1st harmonic of slider-crank motion
F_{max} = maximum inertia force, lb
C = variables defined by equations
K = a constant in the Fourier Series
α = variable defined by equations
u = pressure angle, degrees
R_ψ = maximum output rotation, degrees
ϕ = input angle (drive crank)
ψ = output angle (driven link)
r_p = pitch radius of gear in example, in.

Subscripts, 1, 2, 3 . . . m refer to the order number of the harmonic; for example, mth harmonic; note that m is also used as a factor in the equations. Angular values, shown in radians, can be converted to degrees by multiplying by 57.3.



Crank-and-Rocker Mechanism for High-Speed Operation

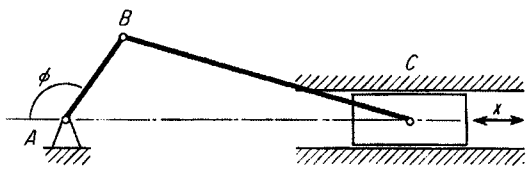
Harmonics of certain crank-and-rocker linkages

m	a_m	b_m	c_m	d_m	f_m
Case 1: u = 5, Rψ = 7.073315					
1	0.043660	−0.043641	0.061733	1.000000	0.000000
2	−0.000953	0.000000	0.000953	0.015437	0.015427
3	0.000028	−0.000014	0.000031	0.005022	0.000000
4	−0.000001	0.000000	0.000001	0.000012	0.000004
5	.....	.....	.....	.....	.....
6	.....	.....	.....	.....	.....
Case 2: u = 10, Rψ = 14.160203					
1	0.087490	−0.087131	0.123479	1.000000	0.000000
2	−0.003827	0.000000	0.003827	0.030993	0.030944
3	0.000223	−0.000111	0.000249	0.002017	0.000000
4	−0.000015	.....	0.000015	0.000121	0.000030
5	0.000001	.....	0.000001	0.000008	.....
6	.....	.....	.....	.....	.....
Case 3: u = 15, Rψ = 21.274683					
1	0.131650	−0.130551	0.185405	1.000000	0.000000
2	−0.008666	0.000000	0.008666	0.046741	0.046531
3	0.000760	−0.000375	0.001000	0.005394	0.000000
4	−0.000075	0.000000	0.000075	0.000405	0.000101
5	0.000008	−0.000003	0.000008	0.000043	.....
6	−0.000001	0.000000	0.000001	0.000005	.....
Case 4: u = 20, Rψ = 28.431708					
1	0.176330	−0.173616	0.247458	1.000000	0.000000
2	−0.015546	0.000000	0.015546	0.062823	0.062318
3	0.001827	−0.000892	0.002000	0.008082	0.000000
4	−0.000242	0.000000	0.000242	0.000978	0.000242
5	0.000034	−0.000012	0.000036	0.000145	0.000000
6	−0.000005	0.000000	0.000005	0.000020	0.000002
Case 5: u = 25, Rψ = 35.647729					
1	0.221690	−0.216365	0.309774	1.000000	0.000000
2	−0.024573	0.000000	0.024573	0.079326	0.078367
3	0.003632	−0.001750	0.004000	0.012913	0.000000
4	−0.000604	0.000000	0.000604	0.001950	0.000479
5	0.000107	−0.000038	0.000114	0.000368	0.000000
6	−0.000020	0.000000	0.000020	0.000065	0.000005
Case 6: u = 30, Rψ = 42.941404					
1	0.267950	−0.258654	0.372423	1.000000	0.000000
2	−0.035898	0.000000	0.035898	0.096390	0.094741
3	0.006413	−0.003037	0.007071	0.018987	0.000000
4	−0.001289	0.000000	0.001289	0.003461	0.000843
5	0.000276	−0.000096	0.000292	0.000784	0.000000
6	−0.000062	0.000000	0.000062	0.000166	0.000013

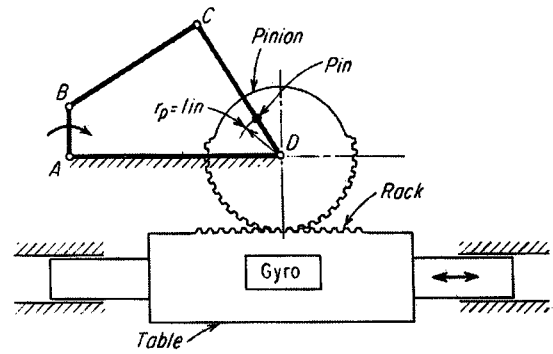
m	a_m	b_m	c_m	d_m	f_m
Case 7: u = 35, Rψ = 50.334529					
1	0.315300	−0.300351	0.435460	1.000000	0.000000
2	−0.049707	0.000000	0.049707	0.114153	0.111509
3	0.010448	−0.004843	0.011489	0.026384	0.000000
4	−0.002471	0.000000	0.002471	0.005674	0.001364
5	0.000623	−0.000210	0.000657	0.001509	0.000000
6	−0.000164	0.000000	0.000164	0.000377	0.000027
Case 8: u = 40, Rψ = 57.853304					
1	0.363970	−0.341265	0.498935	1.000000	0.000000
2	−0.066237	0.000000	0.066237	0.132757	0.128740
3	0.016072	−0.007244	0.017607	0.035289	0.000000
4	−0.004387	0.000000	0.004387	0.008793	0.002077
5	0.001277	−0.000408	0.001414	0.002834	0.000000
6	−0.000388	0.000000	0.000388	0.000778	0.000052
Case 9: u = 45, Rψ = 65.530207					
1	0.414210	−0.381196	0.562920	1.000000	0.000000
2	−0.085785	0.000000	0.085785	0.152392	0.146547
3	0.023689	−0.010303	0.025826	0.045879	0.000000
4	−0.007359	0.000000	0.007359	0.013073	0.003019
5	0.002439	−0.000724	0.002646	0.004700	0.000000
6	−0.000842	0.000000	0.000842	0.001496	0.000091
Case 10: u = 50, Rψ = 73.406779					
1	0.466310	−0.419839	0.627463	1.000000	0.000000
2	−0.108723	0.000000	0.108723	0.173274	0.164957
3	0.033799	−0.014039	0.036592	0.058318	0.000000
4	−0.011821	0.000000	0.011821	0.018839	0.004227
5	0.004410	−0.001190	0.004472	0.007127	0.000000
6	−0.001714	0.000000	0.001714	0.002732	0.000149
Case 11: u = 55, Rψ = 81.538039					
1	0.520570	−0.456870	0.692620	1.000000	0.000000
2	−0.135496	0.000000	0.135496	0.195628	0.184075
3	0.047024	−0.018417	0.050498	0.072908	0.000000
4	−0.018359	0.000000	0.018359	0.026507	0.005737
5	0.007646	−0.001824	0.007810	0.011277	0.000000
6	−0.003317	0.000000	0.003317	0.004789	0.000232
Case 12: u = 60, Rψ = 90.00000					
1	0.577350	−0.491515	0.758235	1.000000	0.000000
2	−0.166667	0.000000	0.166667	0.219810	0.204012
3	0.064150	−0.023280	0.068242	0.090001	0.000000
4	−0.027778	0.000000	0.027778	0.036635	0.007592
5	0.012830	−0.002619	0.013115	0.017890	0.000000
6	−0.006173	0.000000	0.006173	0.008141	0.000345

PROPORTIONS OF CRANK-AND-ROCKER LINKAGES

Case	u	AB/AD	BC/AD = CD/AD
1	5	0.04366	0.707836
2	10	0.08749	0.709887
3	15	0.13165	0.713282
4	20	0.17633	0.718091
5	25	0.22169	0.724385
6	30	0.26795	0.732164
7	35	0.31530	0.741499
8	40	0.36397	0.752602
9	45	0.41421	0.765473
10	50	0.46631	0.780324
11	55	0.52057	0.797297
12	60	0.57735	0.816604



In-Line Slider Crank



Crank-and-Rocker For Gyro Shake Table

minus 90°. The ideal pressure angle is zero, but in practice it is held to within a prescribed limit. Values of u from 5 to 60° in increments of 5 are given in the table with the corresponding values of R_ψ, the maximum output rotation in degrees.

The other terms in the working equation are defined by the following mathematical relationships, and they determine the various amplitudes and ratios shown in the table.

Equation for 1st harmonic

$\alpha = \sin u$; $\alpha_2 = \frac{1}{4}\alpha^2$; $\alpha_4 = 3/64\ \alpha^4$; $\alpha_6 = 5/512\ \alpha^6$; $\alpha_8 = 35/(128)^2\ \alpha^8$

$C_8 = \alpha_8 \ldots$
$C_6 = \alpha_6 + 8C_8 \ldots$
$C_4 = \alpha_4 + 6C_6 - 20C_8 \ldots$
$C_2 = \alpha_2 + 4C_4 - 9C_6 + 16C_8 \ldots$
$C_o = 1 + C_2 - C_4 + C_6 - C_8 \ldots$
$C_m = (m \text{ odd}) = 0$

The working equation for output angle can be expressed in the following forms:

$$\psi = K + \sum_{m=1}^{\infty} (a_m \sin m\phi + b_m \cos m\phi)$$

$$\psi = K + \sum_{m=1}^{\infty} c_m \sin (m\phi + \phi_m)$$

which give us the following:

$$\psi = K + c_1 \sum_{m=1}^{\infty} d_m \sin (m\phi + \phi_m)$$

where

$a_m = (-\tan\frac{1}{2}u)^m/(-\mathrm{m})$
$b_1 = -\frac{1}{4}\sin u\ (2C_o - C_2)$
$b_m\ (m \text{ even}) = 0$

$$b_m\ (m \text{ odd}) = \frac{-\sin u}{4m} = (C_{m-1} - C_{m+1})$$

$c_m = \sqrt{a^2_m + b^2_m}$; $\cos \phi_m = a_m/c_m$
$d_m = c_m/c_1$

In-line slider-crank mechanism

Equation above for the first harmonic can be compared with the corresponding equation in the harmonic analysis of an in-line slider-crank mechanism (diagram above) taken from *Engineering Dynamics—Vol IV—Internal Combustion Engines*, by C. B. Biezeno and R. Grammel, translated by M. P. White, Blackie and Son Ltd, London, p 6, Eq (5) and (6):

$$X = X_o + e_1 \sum_{m=1}^{\infty} f_m \sin m\phi$$

where X is the slider displacement and e_1 and f_m correspond to c_1 and d_m in the equation for the first harmonic.

The value of the link length ratio AB/BC for the in-line

HARMONIC ANALYSIS OF HIGH-SPEED MACHINERY

■ What is meant by "harmonic analysis"? It covers determination of resonant speeds, the severity of passing through various speed ranges (including resonant speed), and magnitude of the inertia forces. Such analysis will tell you how to design for efficient operation at high speeds and how to avoid undue resonance disturbances.

To determine these factors it is useful to express the motion in a Fourier Series—in other words, to express it in a series of harmonics of the form:

$Y = K + a_1 \sin(\omega t + \phi_1) + a_2 \sin(2\omega t + \phi_2) + \cdots + a_m \sin(m\omega t + \phi_m) + \cdots$

Here, Y represents the motion to be analyzed, K = a constant, a_m = amplitude of mth harmonic, ω = angular velocity, t = time—so that ωt represents ϕ, an angular displacement as a function of time. The term $a_1 \sin(\omega t + \phi_1)$ is the first harmonic or fundamental The other terms are the higher harmonics (like overtones in the musical field.

The harmonics of a given motion are used in the determination of the significance of resonant speeds and the corresponding inertia forces, which are proportional to the second derivative of Y. Higher harmonics are often neglected, as sometimes in connecting-rod motion of internal combustion engines. For complex motions as in crank-and-rocker mechanisms, the higher harmonics must be evaluated because the resulting amplitudes can be of greater significance. In these higher harmonics the associated critical speed is usually taken to be $m\omega t$, radians/sec. In some high-speed cam applications, harmonics as high as the 20th have been considered significant. ■

slider-crank should be identical to the link length ratio AB/BC for the crank-and-rocker. On this basis the tabular data include values for d_m and f_m which can be used to compare these ratios for the two types of mechanisms.

For small values of the maximum pressure angle u, the in-line slider-crank has practically the same value of harmonic amplitudes as the crank-and-rocker for even harmonics; but the slider-crank values are slightly lower.

As pressure angle increases, the slider-crank values become progressively lower than crank-and-rocker values. Thus, at a pressure angle of 30°, d_2 and f_2 differ by approximately 1.7%; while at 60° the difference is approximately 8%.

The ratio of 2nd to 1st harmonic amplitude, and 3rd to 1st harmonic amplitude can be significant in high-speed operation, as the following ratios show:

$d_2\,(30°) = 9.639\%$ $\qquad$ $d_2\,(60°) = 21.981\%$
$d_3\,(30°) = 1.90\%$ $\qquad$ $d_3\,(60°) = 9.0\%$

As a guide, it would seem reasonable to limit the maximum pressure angle to 30° or less in high-speed applications, and to 45° or less for moderate speeds; also 60° usually can be useful eventually only in low-speed and light-duty instrument drives.

The range R_ψ is approximately proportional to the pressure angle with the "constant" of proportionality varying between 1.41 and 1.50. The ratios d_m, c_m, amplitudes and the link length ratio AB/AD also appear to be approximately linear with the pressure angle—at least sufficiently so for qualitative estimates. Accurate values, however, can be calculated from the table.

Example

Analyze the harmonics of a shake table (sketch, p 49) for testing gyroscopes and other devices. The proportions of the crank-and-rocker are selected for high-speed operation with maximum pressure angle of 25°, the input crank rotates at 10 rps, and the gyro weighs 8 oz. Find:

(1) Proportions of the linkage;

(2) Inertia force of the gyro due to the 2nd harmonic, compared with the 1st harmonic;

(3) Amplitude of the first three harmonics of the table motion.

SOLUTION

(1) From the table, a 25° pressure angle corresponds to Case 5. The proportions are $AB/AD = 0.22169$; $BC/ADCD/AD = 0.724385$. If fixed link AD is 10 in., the driving crank AB is 2.217 in.; the coupler BC and the driven link CD are 7.244 in. each.

(2) The inertia force is equal to the mass of the gyro multiplied by its acceleration, and gyro acceleration is equal to pitch radius r_p of the gear multiplied by its angular acceleration. For the mth harmonic the gyro acceleration is

$$A_G = r_p \frac{d^2}{dt^2}\left[c_m \sin\,(m\phi + \phi_m)\right]$$

$$A_G = r_p\, m^2\, c_m \left(\frac{d\phi}{dt}\right)^2 \sin\,(m\phi + \phi_m)$$

Therefore maximum inertia force F_{max} due to mth harmonic is:

$$F_{max} = \left[\frac{\text{gyro weight (lb)}}{g\left(\frac{\text{in}}{\text{sec}^2}\right)}\right] \times \left[r_p\, m^2\, c_m \left(\frac{d\phi}{dt}\right)^2\right] \text{lb.}$$

If $g = 386$ in./sec^2, gyro weight $= 0.5$ lb, $r_p = 1$ in.,

$$\left(\frac{d\phi}{dt}\right) = 10 \times 2\pi \frac{\text{rad}}{\text{sec.}}$$

$$F_{max} = \frac{0.5}{386} \times 1 \times (20\pi)^2 \times m^2\, c_m$$

Taking m and c_m from the table gives:

1st harmonic $m = 1$; $c_m = 0.309774$; $F_{max} = 1.587$ lb.
2nd harmonic $m = 2$; $c_m = 0.024573$; $F_{max} = 0.503$ lb

$$\text{Ratio 2nd to 1st harmonic} = \frac{0.503}{1.587} \times 100 = 31.7\%$$

(3) From the table for Case 5, the amplitudes of the first harmonics are given in the c_m column as:

$c_1 = 0.309774$; $c_2 = 0.023573$; $c_3 = 0.00400$ radians

Ratio of higher harmonics to the 1st harmonic can also be obtained from the d_m column. For example,

Ratio of amplitude of 2nd to 1st: $(d_2) = 0.079326 = 7.9\%$

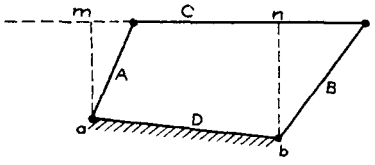
Ratio of amplitude of 3rd to 1st: $(d_3) = 0.012913 = 1.3\%$

The significance of these values depends upon the particular design requirements, as well as the number of decimal places you want to use.

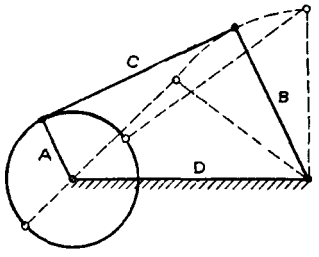
ACKNOWLEDGEMENTS: The authors are indebted to the Electronic Research Laboratories of Columbia University for having made available their computational facilities and to Mr. Sy Newman for programing help on the IBM 650 computer.

Four-Bar Linkages and Typical Industrial Applications

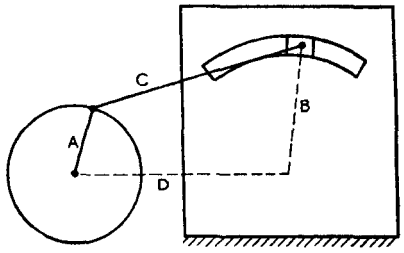
All mechanisms can be broken down into equivalent four-bar linkages. They can be thought of as the basic mechanisms and are useful in many mechanical operations.



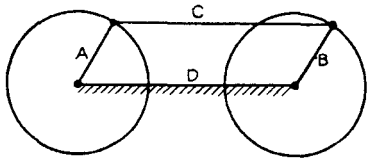
FOUR-BAR LINKAGE—Two cranks, a connecting rod and a line between the fixed centers of the cranks make up the basic four-bar linkage. Cranks can rotated if A is smaller than B or C or D. Link motion can be predicted.



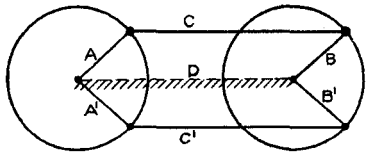
CRANK AND ROCKER — Following relations must hold for operation: $A+B+C>D$; $A+D+B>C$; $A+C-B<D$, and $C-A+B>D$.



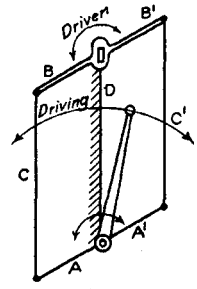
FOUR-BAR LINK WITH SLIDING MEMBER—One crank replaced by circular slot with effective crank distance of B.



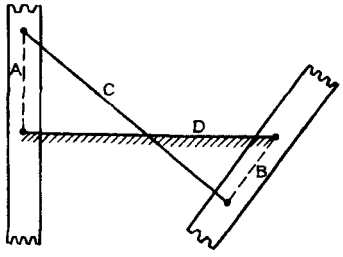
PARALLEL CRANK FOUR-BAR—Both cranks of the parallel crank four-bar linkage always turn at the same angular speed but they have two positions where the crank cannot be effective. They are used on locomotive drivers.



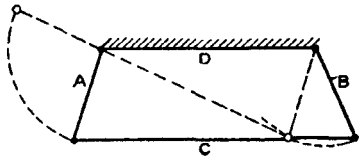
DOUBLE PARALLEL CRANK—This mechanism avoids dead center position by having two sets of cranks at 90 deg advancement. Connecting rods are always parallel. Sometimes used on driving wheels of locomotives.



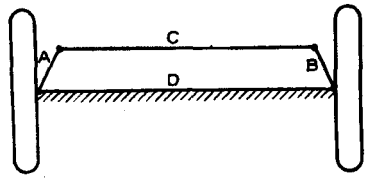
PARALLEL CRANKS—Steam control linkage assures equal valve openings.



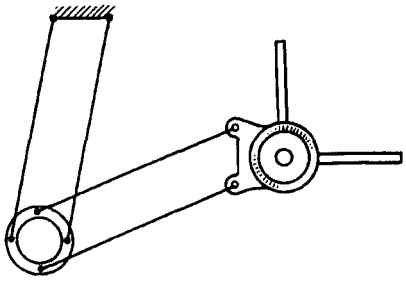
NON-PARALLEL EQUAL CRANK—The centrodes are formed as gears for passing dead center and can replace ellipticals.



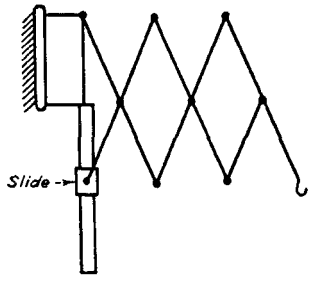
SLOW MOTION LINK—As crank A is rotated upward it imparts motion to crank B. When A reaches dead center position, the angular velocity of crank B decreases to zero. This mechanism is used on the Corliss valve.



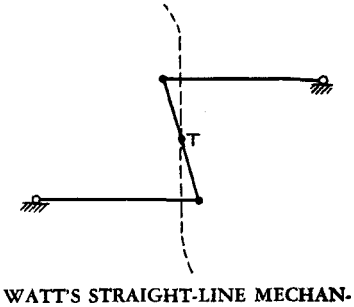
TRAPAZOIDAL LINKAGE—This linkage is not used for complete rotation but can be used for special control. Inside moves through larger angle than outside with normals intersecting on extension of rear axle in cars.



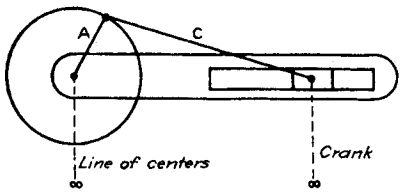
DOUBLE PARALLEL CRANK MECHANISM—This mechanism forms the basis for the universal drafting machine.



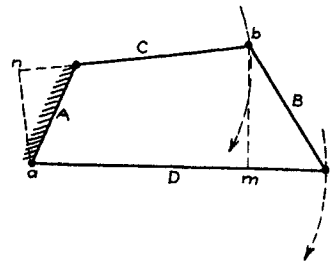
ISOSCELES DRAG LINKS—"Lazy-Tong" device made of several isosceles links; used for movable lamp support.



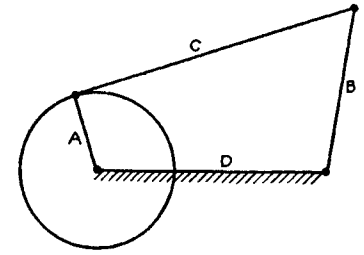
WATT'S STRAIGHT-LINE MECHANISM—Point T describes line perpendicular to parallel position of cranks.



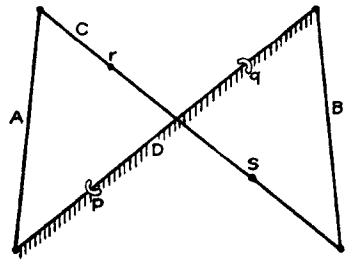
STRAIGHT SLIDING LINK—This is the form in which a slide is usually used to replace a link. The line of centers and the crank *B* are both of infinite length.



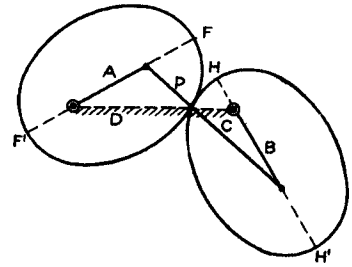
DRAG LINK—This linkage used as the drive for slotter machines. For complete rotation: $B>A+D-C$ and $B<D+C-A$.



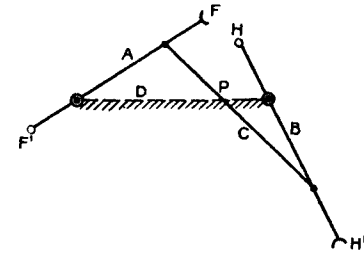
ROTATING CRANK MECHANISM—This linkage is frequently used to change a rotary motion to swinging movement.



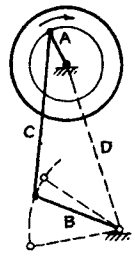
NON-PARALLEL EQUAL CRANK—If crank *A* has uniform angular speed, *B* will vary.



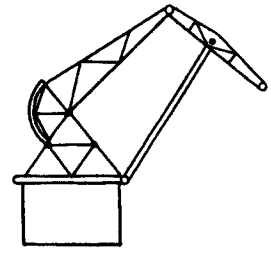
ELLIPTICAL GEARS—They produce same motion as non-parallel equal cranks.



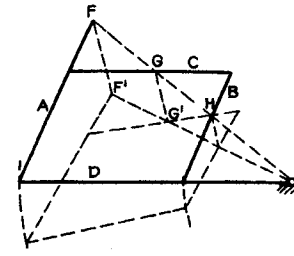
NON-PARALLEL EQUAL CRANK—Same as first but with crossover points on link ends.



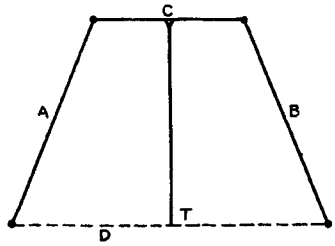
TREADLE DRIVE—This four-bar linkage is used in driving grindwheels and sewing machines.



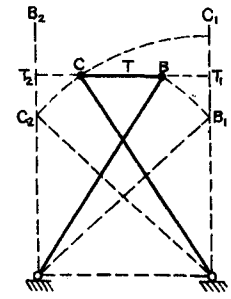
DOUBLE LEVER MECHANISM—Slewing crane can move load in horizontal direction by using D-shaped portion of top curve.



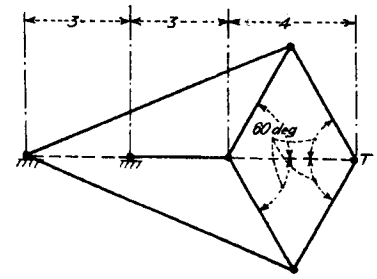
PANTOGRAPH—The pantograph is a parallelogram in which lines through *F*, *G* and *H* must always intersect at a common point.



ROBERT'S STRAIGHT-LINE MECHANISM—The lengths of cranks *A* and *B* should not be less than 0.6 *D*; *C* is one half *D*.



TCHEBICHEFF'S—Links made in proportion: $AB=CD=20$, $AD=16$, $BC=8$.



PEUCELLIER'S CELL—When proportioned as shown, the tracing point *T* forms a straight line perpendicular to the axis.

Four-Bar Power Linkages

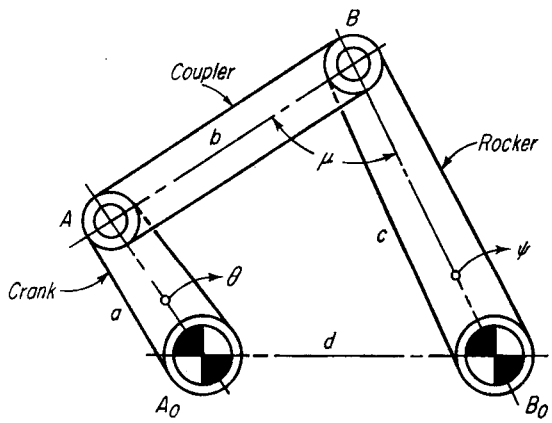
With large forces or high speeds, the transmission angle becomes highly important. These new German charts simplify design.

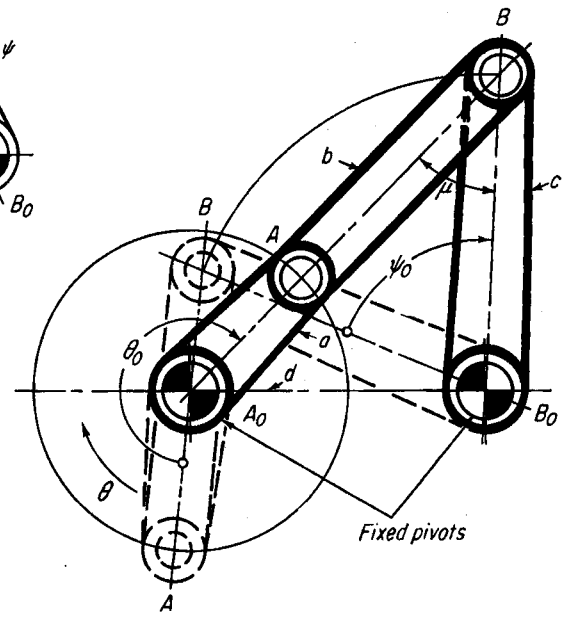
J. Volmer & Preben W. Jensen

THE most common function of a four-bar linkage is transforming rotary motion into oscillating motion. Frequently, in such applications, a large force must be transmitted, or force must be converted at high speed. it is then that a factor called the *force-transmission angle* becomes of paramount importance.

The force-transmission angle, angle μ in Fig 1, is comparable to the pressure angle in cams. For best results, μ should be as close to 90 deg as possible during the entire rotation of the crank. This will reduce bending in the linkages and will produce the most favorable force-transmission conditions. (When μ becomes small, a large force is required to drive the rocker arm, and the force fluctuations increase.)

The charts presented here make it easier to find the best force-transmission linkage in a wide range of possible selections. Four examples show how to apply the charts to a number.

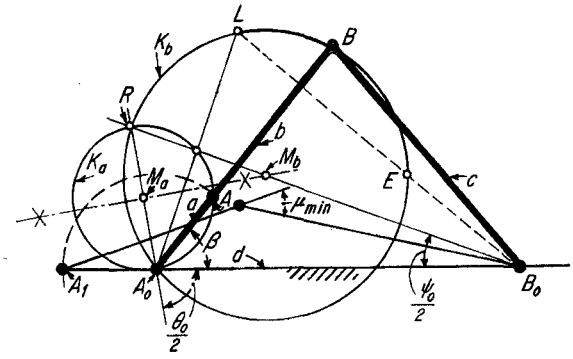




1. Four-bar linkage, (above) when operating as power-driving crank-and-rocker mechanism, should be designed with the force-transmission angle μ as close to 90 deg as possible. Sketch at right shows its two dead-center positions with input and output requirements defined by θ_0 and ψ_0, respectively.

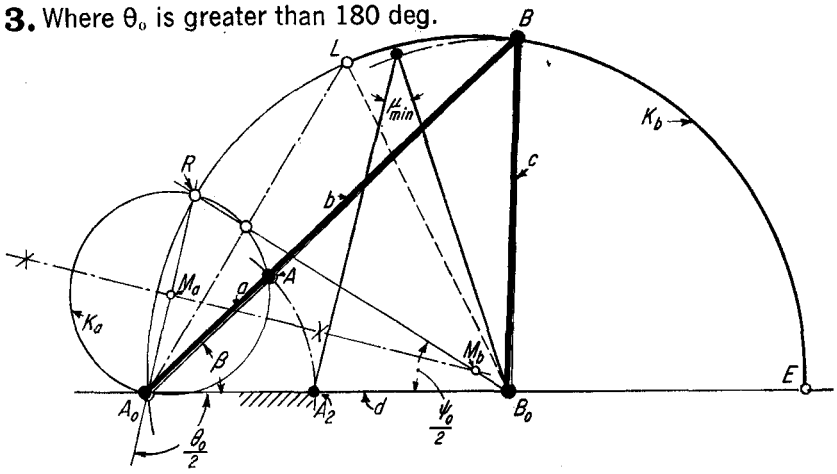
Layout techniques

For finding family of linkages which meet θ_0 and ψ_0 requirements. Pin B of the linkages will lie somewhere along arc LE — but only one linkage of each family will have optimum transmission angle.



2. Where θ_0 is less than 180 deg.

3. Where θ_0 is greater than 180 deg.



The linkage family

When a four-bar power linkage is designed to operate, for example, as a quick-return mechanism, angular displacement of the output link (the rocker) is usually prescribed in relationship to that of the input link (the crank). These angles are measured from the dead-center positions of the rocker (where the rocker reverses its direction of rotation). Thus, in Fig 1, ψ_0 is the angle between the two dead-center positions of the rocker, and θ_0 is the corresponding angle of the crank. Both angles are measured clockwise from the *inside* dead-center position, where crank and coupler are superimposed, to the outside dead-center position.

When values are given for ψ_0 and θ_0, and for the distance between centers, A_0B_0, a family of linkages that meets these conditions can be evolved by a graphical method developed some years ago by H. Alt. His method, illustrated in Fig 2, is the first phase of the design procedure:

1) Lay out the line of centers, A_0B_0.

2) Construct angles $\theta_0/2$ with center at A_0 and $\psi_0/2$ with center at B_0. Both angles are measured in the same direction. This locates point R.

3) Draw M_aM_b, the perpendicular bisector of A_0R. Locate point M_b.

4) Draw circles K_a and K_b through point R with M_a and M_b as centers, respectively.

5) On circle K_b make $RL = A_0R$ and connect point L with B_0 to locate point E on circle K_b.

6) Choose any point on the circular arc LE as point B, which is the center of the moving joint. This will be one of a family of linkages.

7) Construct line A_0B to intersect circle K_a. This locates crank pivot point A, thus defining all dimensions of the required four-bar linkage.

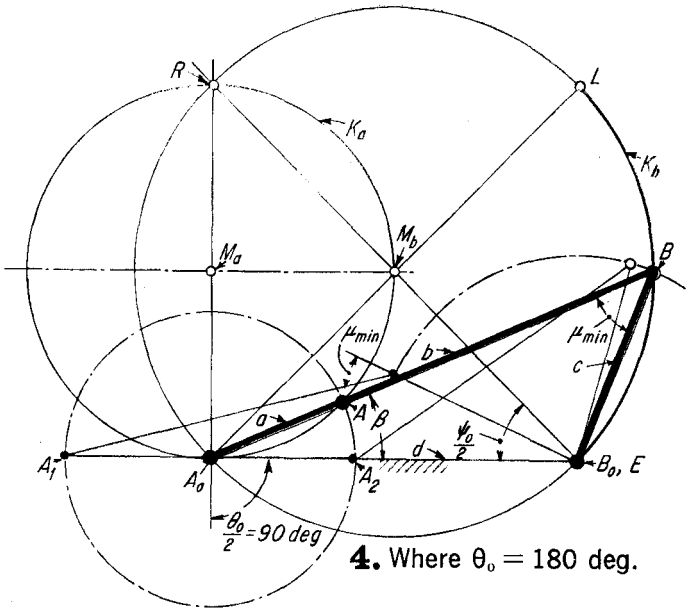
This method, however, stops short of determining which of all possible mechanisms obtained from the construction is the best power linkage.

Transmission angles

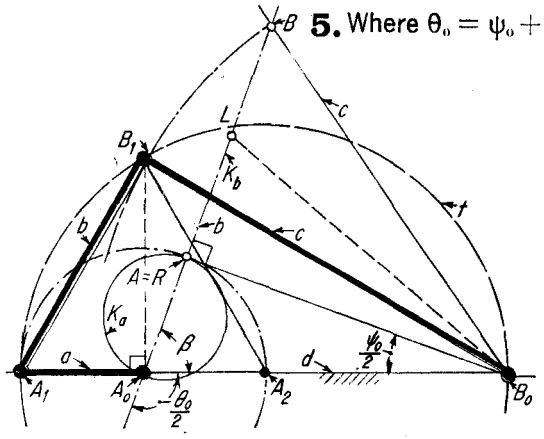
The question is, which linkage, from the above family, has μ_{min} closest to 90 deg. Since angle μ can be either smaller or larger than 90 deg, it is useful to define it as the angle between AB and BB_0—or between the extension of line AB and BB_0. Thus μ is always taken as $\mu \leqq 90$ deg. For example, if $\mu \simeq 120$ deg, it is taken as $\mu = 60$ deg; a linkage in which μ varies from 75 deg to 120 deg ($\mu_{min} = 60$ deg) is more desirable than a linkage where μ varies from 45 deg to 90 deg ($\mu_{min} = 45$ deg).

The design charts

Previous investigations carried out by the first-mentioned author, J. Vol-



4. Where $\theta_0 = 180$ deg.



5. Where $\theta_0 = \psi_0 + 180$ deg.

mer, have shown how to find the crank positions where μ is minimum. For $\theta_0 < 180$ deg, μ_{min} occurs when crank pin A is at A_1 (see Fig 2); for $\theta_0 > 180$ deg when A is at A_2, see Fig 3; when $\theta_0 = 180$ deg the minima of μ at A_1 and A_2 are equal (see Fig 4). The linkage with $\theta_0 = 180$ deg is called a centric crank-and-rocker mechanism.

Recently the second author, Preben W. Jensen, showed that among the family of linkages with given angles θ_0 and ψ_0, the optimum linkage can be determined with the aid of a chart originally published by H. Alt in *Verein Deutscher Ingenieure*, Vol 85 (1941) p 69. Alt's chart, however, was not very exact. It has now been corrected and also completed for the whole range of angles θ_0 and ψ_0. The chart is shown in Fig 7.

Here is how to use the chart to find the linkage with the largest μ_{min} value from the linkage family in Fig 2:

For the given values of θ_0 and ψ_0, find the value for angle β from the chart (dashed lines). Angle β = angle AA_0B_0. This locates point B and thus defines the mechanism. Read also the value for max μ_{min} (solid lines). No better minimum transmission angle can be obtained for the given conditions—every other linkage for the same dead-center angles possess a lower value of μ_{min}.

Interpolation can be employed between the curves for β and μ_{min}. However, when more accurate values for β are needed, use the equations (by assuming values for β and d):

$$\frac{a}{d} = -\frac{\sin\frac{\psi_0}{2}\cos\left(\frac{\theta_0}{2}+\beta\right)}{\sin\left(\frac{\theta_0}{2}-\frac{\psi_0}{2}\right)}$$

$$\frac{b}{d} = \frac{\sin\frac{\psi_0}{2}\sin\left(\frac{\theta_0}{2}+\beta\right)}{\cos\left(\frac{\theta_0}{2}-\frac{\psi_0}{2}\right)}$$

$$c^2 = (a+b)^2+d^2-2(a+b)d\cos\beta$$

where $a = AA_0$, $b = AB$, $c = BB_0$, and $d = A_0B_0$.

The design chart in Fig 6 is for the special case where $\theta_0 = 180$ deg, and the chart in Fig 9 for the design of a slider crank mechanism. The following examples illustrate use of the three charts.

Example I—Typical design

The dead-center position construction for $\theta_0 = 160$ deg and $\psi_0 = 40$ deg is shown in Fig 2. From the chart in Fig 7 find $\beta = 50.5$ deg and max $\mu_{min} = 32$ deg. Although there is no

linkage with more favorable force transmission characteristics, a linkage with such a low minimum value of transmission angle is not capable of running at high speeds or transmitting great forces.

Example II—When $\theta_0 = \psi_0 + 180$ deg

When θ_0 and ψ_0 are chosen so that $\theta_0 = \psi_0 + 180$ deg at the dead-center position construction, then angle A_0RB_0 becomes a right angle, and circle K_b degenerates into the line A_0R. Fig 5 shows the construction for $\theta_0 = 220$ deg and $\psi_0 = 40$ deg. For all linkages within this family, the crank-pin center A coincides with point R, and line A_0R extended beyond point L is the locus of point B, where $A_0R = RL$. Therefore, angle β is useless for finding the optimum linkage.

To obtain this linkage, locate A, (A_1 $A_0 = A_0A$). Draw circle t, with A_1B_0 as the diameter, to intersect with a line from A_0 perpendicular *to* A_0B_0. This locates point B_1. The required optimum linkage is $A_0A_1B_1B_0$. Angle $A_2B_1B_0 = \max \mu_{min}$. All mechanisms of this family lie on the diagonal dashed line in the chart, Fig 7. For this example, the chart shows that max $\mu_{min} = 30$ deg, and that $\beta = 70$ deg.

Example III—Centric linkage design

The centric four-bar linkage where the crank angle rotates 180 deg between the two dead center positions ($\theta_0 = 180$ deg) provides the greatest amplitudes ψ_0 of the rocker with most favorable transmission angle. However, Fig 7 is not applicable for this case because it indicates that $\beta = 0$ deg for any desired rocker angle ψ_0. This means that the length of crank and rocker must be zero—a solution without practical significance. Thus, a second chart, Fig 6, is used. For example, if a total rocker displacement of $\psi_0 = 90$ deg is required, a mechanism with $\mu_{min} = 40$ deg will be constructed with angle of $\beta = 23$ deg, and a mechanism with $\mu_{min} = 30$ deg with an angle of $\beta = 35$ deg. The dead-center position construction for this mechanism is shown in Fig 4. Here, other conditions can also be prescribed, for instance, a desired length of crank, rocker, or coupler.

Example IV—Slider-crank mechanism

For $\psi_0 = 0$ deg, the four-bar linkage degenerates into the slider-crank mechanism. Instead of the rocker angle ψ_0 the stroke, s, of the slider is used. The construction is as follows (see Fig 8):

1) Draw line A_0B_0 perpendicular to the direction of stroke.

2) Construct angle $\theta_0/2$ and a line parallel to A_0B_0 with distance $s/2$ as shown. This locates point R.

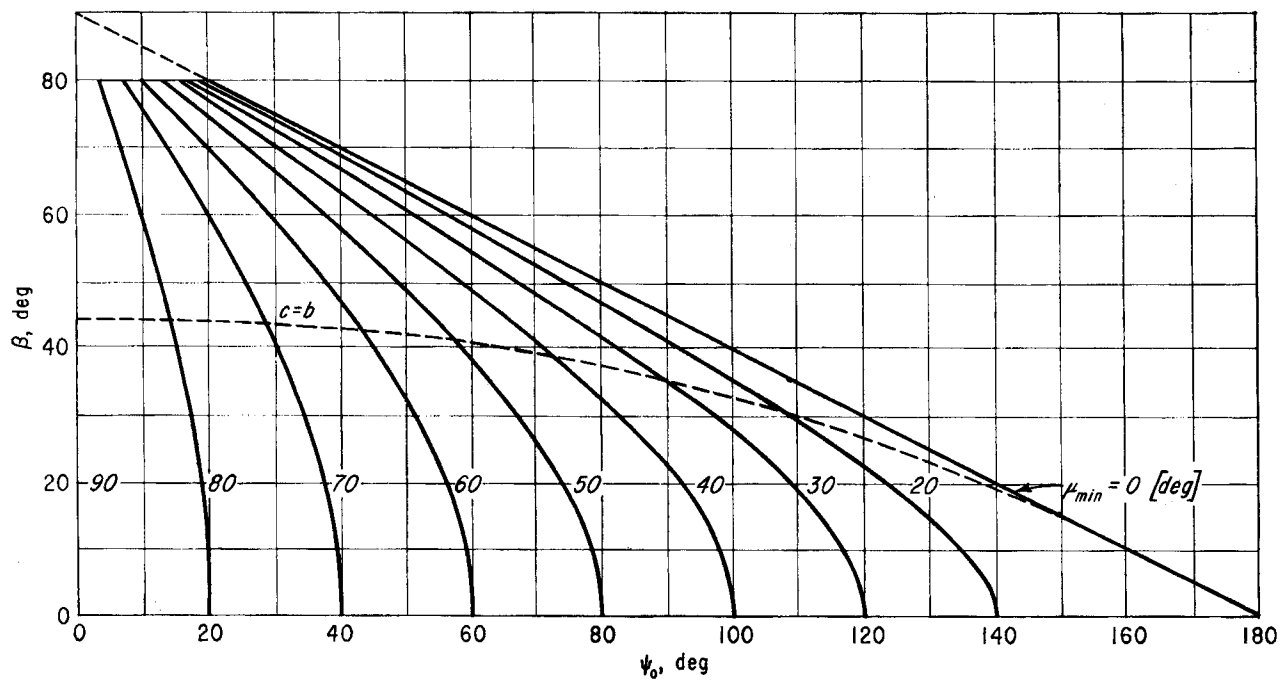
3) Draw M_aM_b, the perpendicular bisector of A_0R.

4) Draw, through R, circle K_a and K_b about M_a and M_b as centers.

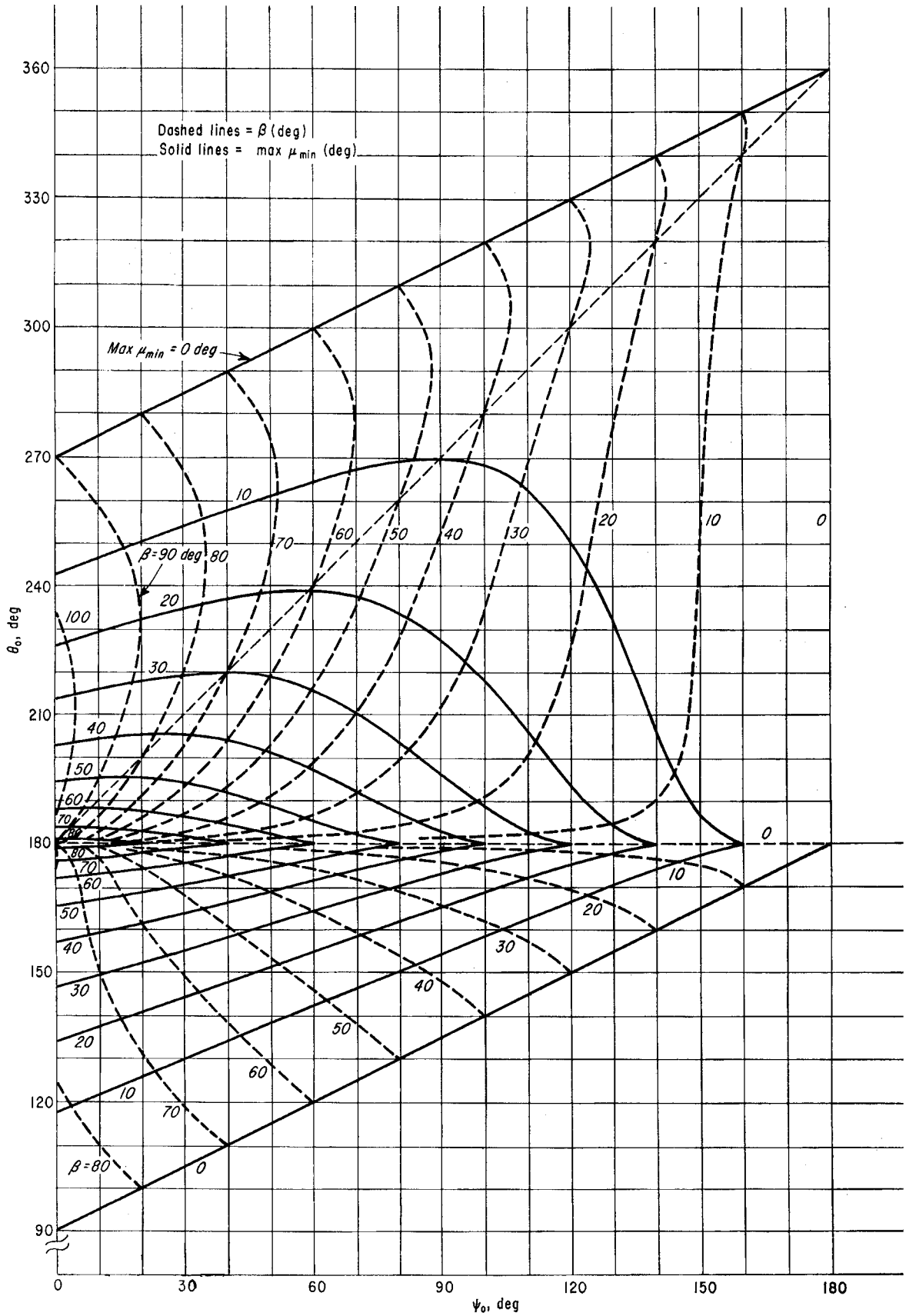
5) The line parallel to A_0B_0 at distance s intersects circle K_b at points L and E. Arc LE on circle K_b is the locus of point B. Point A is on circle K_a in line with A_0B.

For example, suppose a ratio of 4:5 of the crank angles is required for a quick-return mechanism. This ratio provides a dead-center angle of $\theta_0 = 160$ deg and $\theta'_0 = 200$ deg (160:200 = 4:5). As in the case of the crank-and-rocker mechanism, there is a whole family of slider-crank mechanisms fulfilling this condition. Here, the optimum mechanism will be the one whose transmission angle is closest to 90 deg when the crank is perpendicular to the direction of stroke. To find this optimum linkage use the $\psi_0 = 0$ deg ordinate in the chart of Fig 7. Thus, for $\theta_0 = 160$ deg, the chart gives $\beta \approx 76$ deg, which permits completion of the construction in Fig 8 (angle β locates point B on arc LE). The chart also gives max $\mu_{min} = 43$ deg. If the supplementary angle, $\theta'_0 = 200$ deg, is used the construction provides the same linkage because the chart shows $\beta' = 104$ deg, which is the supplement of angle β. (The $c = b$ line in Fig 6 gives points for a linkage with equal coupler and rocker links).

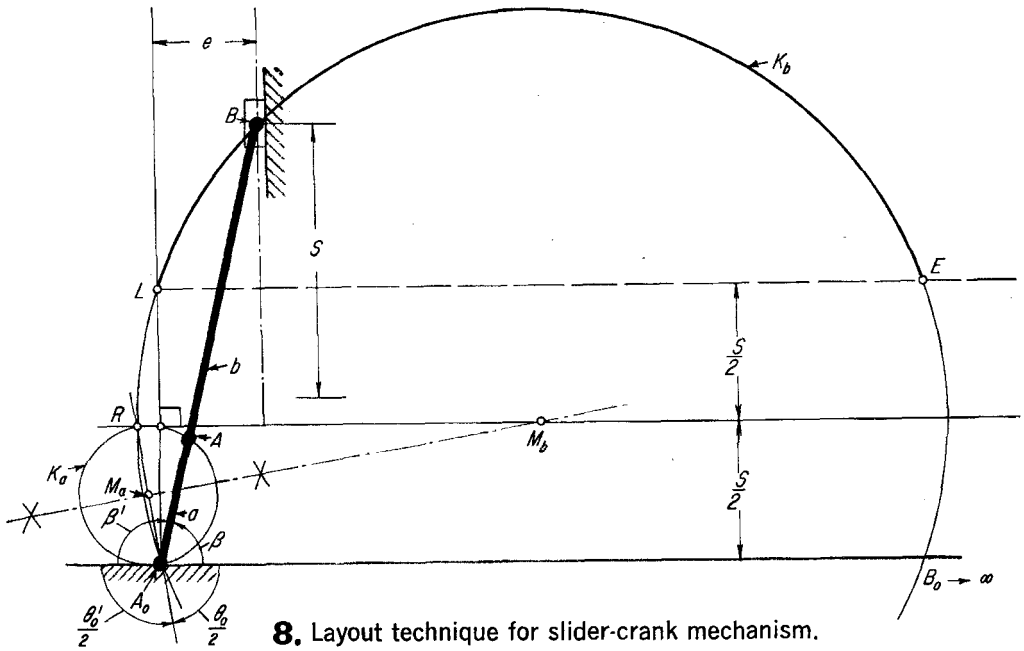
Because an exact value for angle β cannot be found at the ordinate of the chart, a third chart, Fig 9, has



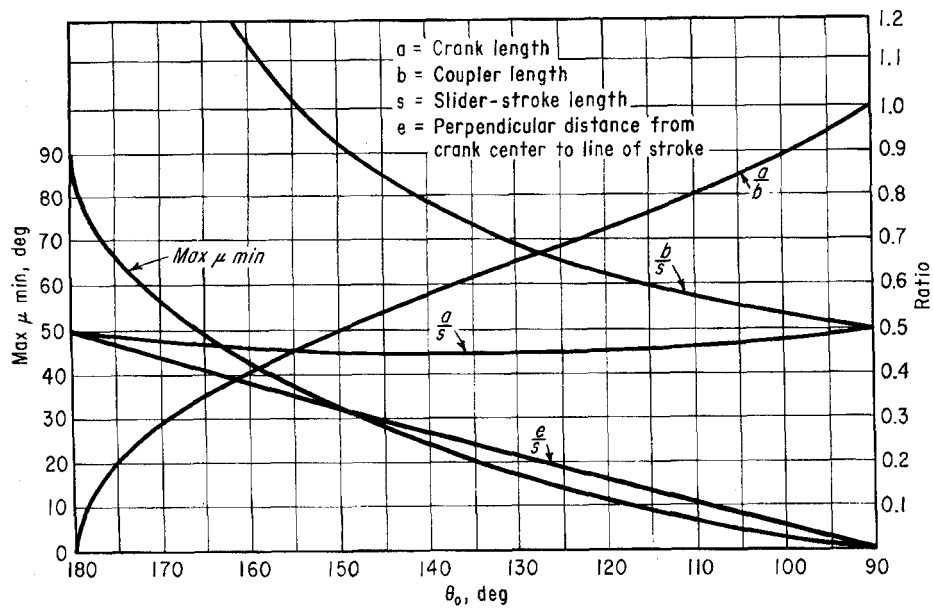
6. Optimum transmission angle — where $\theta_0 = 180$ deg.



7. Optimum transmission angles — general case.



8. Layout technique for slider-crank mechanism.



9. Design chart for slider-crank mechanism.

been developed to provide greater accuracy. Here max μ_{min} is plotted as a function of θ_o, along with ratios a/s, b/s, e/s, and a/b. For the desired angle of $\theta_o = 160$ deg, max $\mu_{min} = 43$ deg, $a/s = 0.465$, $b/s = 1.150$, $a/b = 0.406$, and $e/s = 0.378$. Instead of using angle β in Fig 8, draw a line from point A_o perpendicular to A_oB_o at a distance e—obtained from the ratio e/s—to locate point B on circle K_b. The line connecting A_o and B intersects circle K_a at crank pin center, A, and the optimum slider-crank mechanism for the prescribed ratio of 4:5 is completely defined.

5 Graphic Methods for Designing Four-Bar Linkages

This roundup covers the four-bar linkages that can generate a path approximating a straight line. The methods permit rapid solution of general configurations to fit space requirements, and they simplify mathematical solutions that find precise dimensions.

Erwin P. Pollitt

The classical problem here is to find a four-bar linkage that will generate a straight-line motion. Such linkages are used extensively in mechanical design, in place of cams and gears. Specific applications involve computers, instruments, counterweights and simple translators of rotary motion. Their preference over cams and gears depends upon analysis of their advantages and disadvantages. The advantages are simplicity, negligible friction (and wear), low inertia, economy, and stability in performance. The disadvantages are limited range, approximation of the desired motion, bulkiness, limited control over velocity and acceleration, and the fact that they are relatively difficult to design.

In general, linkages are designed by mathematical or graphical methods. In either case the characteristic problem is that of selecting from a large number of dimensional constants a particular set that will fit the specified space requirements and tolerances. The approach to the problem must be synthetic and approximate, instead of analytical and exact. The five graphical methods presented here offer a practical approach for determining the general configuration. Then, if necessary, mathematical methods can be applied to get the best out of the linkage with respect to the number of precision points and magnitude of deviation from the straight line.

Linkages developed by the five graphical methods have these characteristics:

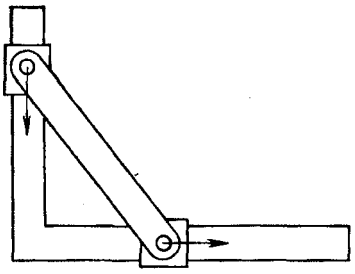
Method A—These give a tracer point with nearly infinite radius of curvature.

Method B—Easiest to construct, but have a limited range characterized by two precision points.

Method C—Relatively harder to construct, but offer greater accuracy—up to five precision points.

Method D—These are the most accurate—up to six precision points. This is the limit because the equation of motion of a point is generally of the sixth order.

Method E—Offer four precision points, but can be adapted to more than four.

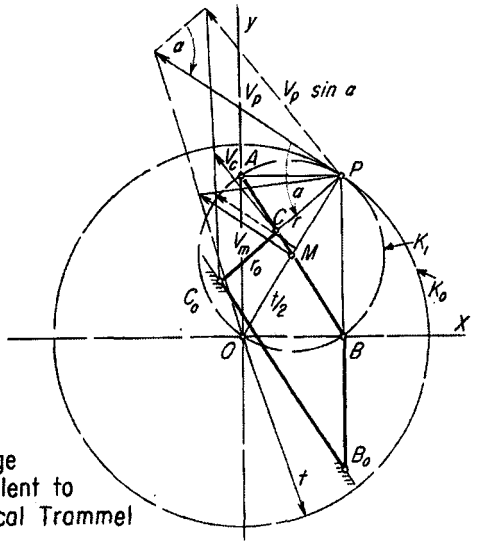


Method A

Synthesis of a Linkage Equivalent to Elliptical Trammel

Approach to solution: **the component of the mechanism containing the point which performs the rectilinear motion is used as the coupler of the linkage; two points on this component are selected as the moving pivots, and the exact—or approximate—center of curvature of their path is made the fixed crank pivot.**

The path of the tracer point of the linkage obtained has an infinite, or nearly infinite radius of curvature.



Linkage Equivalent to Elliptical Trammel

1

The bar is constrained at points A and B to move along the x and y-line. In the linkage the bar is made the coupler. Point B is made one moving pivot, and an arbitrarily chosen point C on the bar the second moving pivot. Crank BB_o, in the design position, is made perpendicular to the x-line, and as long as possible so that B moves approximately along this line while the center of curvature C_o of the point C on the trammel is selected as the second fixed pivot. Point A is then the tracer point.

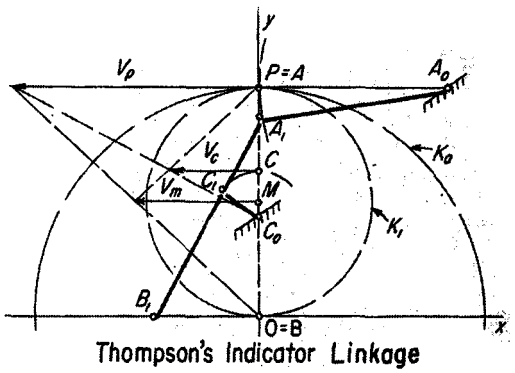
The center of curvature C_o of point C must lie on a line through C and the instantaneous center of rotation P of AB which is found as the intersection of the normals through A and B to y and x respectively.

The radius of curvature CC_o can be calculated, or solved graphically by the use of Euler-Savary's theorem in the form:

$$\frac{V_p \sin \alpha}{V_c} = \frac{r_o}{r_o - r}$$

where V_p is the velocity of the change of position of the instantaneous center, the angle between the tangent to the centrodes at the instantaneous center and the ray C_oP. V_c the velocity of point C of the moving system, $r = CP$, and $r_o = C_oP$.

The velocity of any point of the moving system, for instance, velocity V_m of the midpoint M of AB, is arbitrarily selected, and V_c determined. The fixed centrode of the trammel is the circle K_o with O, the intersection point of x and y, as center, and $OP = t$, the length of bar AB as radius; the moving centrode is the circle K_1 through P and O, with M, the midpoint of AB as center and $t/2$ as radius. The velocity V_p is determined from the consideration that the motion of the bar is equivalent to rolling of the moving centrode on the fixed centrode. V_p is therefore the velocity of this rolling motion, of the point of circle K_1 with which the instantaneous center coincides.

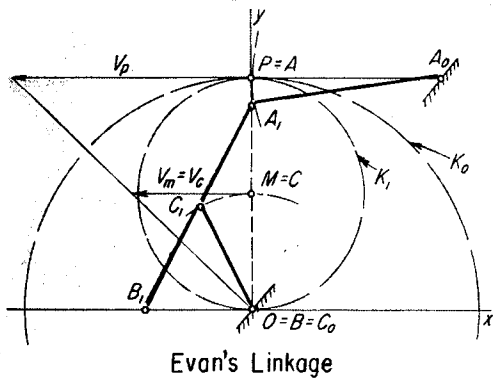


Thompson's Indicator Linkage

2

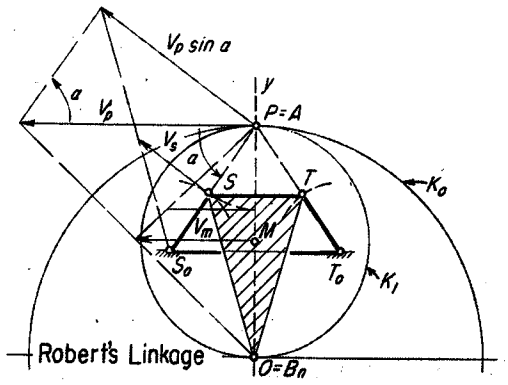
A special case of Fig 1 is the Thompson indicator mechanism shown here. It is widely used in radio sets for indicating the tuned frequency. To determine the linkage, the position of bar AB is chosen so that it coincides with the y-axis; then point A, which coincides with instantaneous center P, is made one moving pivot; and the fixed pivot A_o is located on the perpendicular to y through A. The center C_o of the curvature of the path of an arbitrarily selected moving pivot point C on line AB must also lie on the y-axis. Balance of the procedure is same as Fig. 1.

Having assumed the magnitude of vector V_m of center M of K_1 the vector $V_p \sin \alpha$ can now be constructed. A line drawn through the end points of the vectors V_c and $V_p \sin \alpha$ and the intersection of this line with CP determines point C_o.

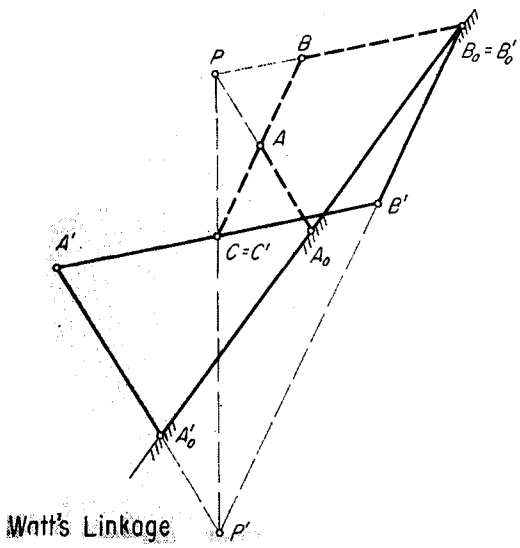


Evan's Linkage

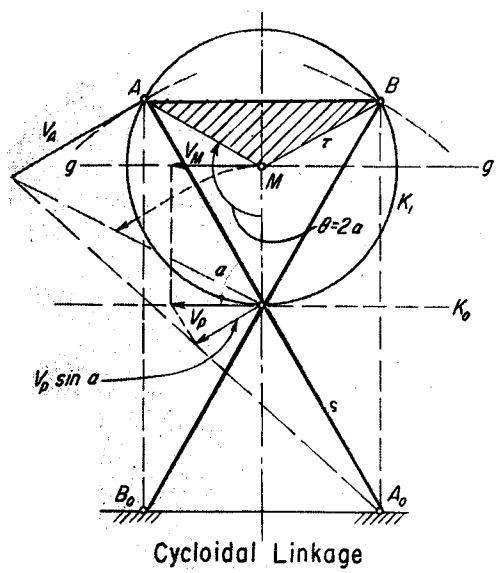
3
This is a second variation where point C coincides with center point M of bar AB; and C_o coincides with points O and B. Here, again, path of A approximates the y-axis, and path of B approximates the x-axis.



4
Here is a third variation of this group, with the moving crank centers symmetrical to AB, which again coincides with the y-axis.

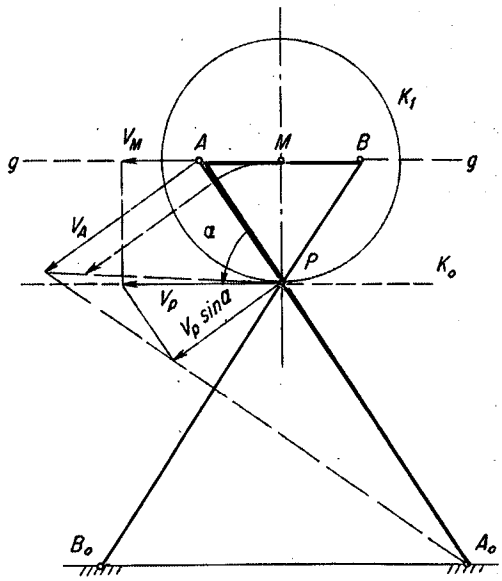


5
By Robert's Law additional linkages can be found such that a point of their coupler performs a path identical to the path of a point of the coupler of a given linkage. In this linkage, A'_oA' and B'_oB' are so related to Evan's linkage that tracer point C' of the coupler of this linkage corresponds to the coinciding tracer point of C of the Evan's coupler. The fixed pivot, B'_o, coincides with B_o, and the fixed link $A'_oB'_o$ has the same direction as the fixed link A_oB_o of the Evan's linkage. Coupler A'B' passes through C and is parallel to BB$_o$, while crank $B'B'_o$ is parallel to AB. Other similarities in the geometry are also apparent. This linkage is similar to Watt's linkage used extensively in steam engines.

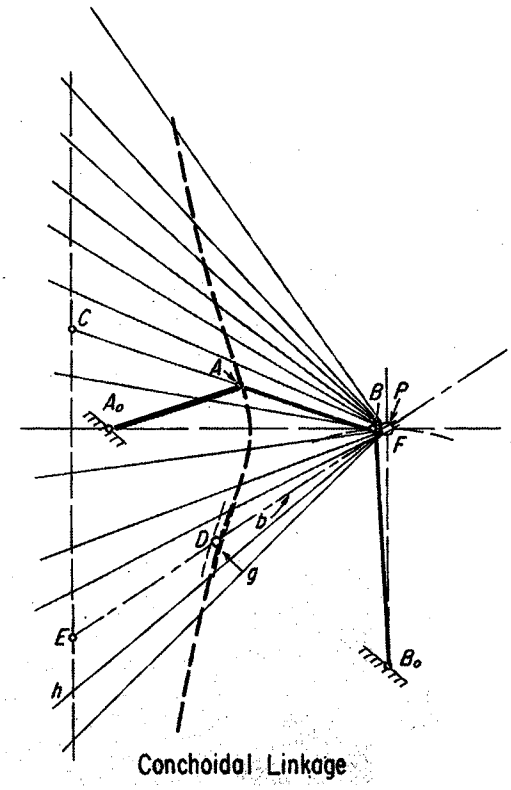


6
Center M of circle K_1 rolling on a straight line K_o generates a straight-line motion along line g-g parallel to K_o. The contact point of the circle K_1 and line K_o is the instantaneous center P. K_o is the fixed centrode and K_1 the moving centrode. The motion of point M is approximated by a coupler link whose tracer point coincides with M, and whose moving pivots AB coincide in the shown position with two points on the periphery of the rolling circle which are symmetrical to the normal line to K_o through M. The centers of curvature, A_o, B_o, of the path of these two points are selected as the fixed pivots of the cranks, and they are determined as previously. The velocity V_p is equal to velocity V_m. The path of points A and B are cycloids whose radius of curvature may be calculated as follows: $\rho = 4r \sin \theta/2$; where r is radius of the rolling circle, and θ is its angle of rotation. The cycloidal linkage is used extensively in the textile industry.

Method A



Tchebishett's Linkage

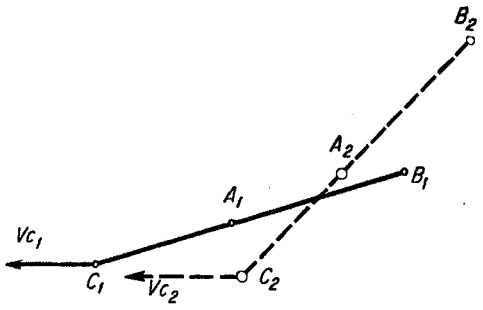


Conchoidal Linkage

7
This is a modification of the previous linkage, with moving crank centers selected so as to coincide with two points inside the periphery located at equal distances from center of the rolling circle. One linkage of this type became known as Tchebisheff's linkage, characterized by a high degree of approximation to straight-line motion of the center point of its coupler.

8
The conchoid is the locus of points D on rays connecting all points of a generating straight line h with a focus F and having equal distance from the straight line. If the motion of a straight bar b is so constrained that the bar passes in all positions through the focus F (by means of a pivoted slide P); and point D, fixed on the bar, moves along a conchoidal groove g, another point, E, of the bar will describe the generating straight line h. Portions of this motion of point E are approximated by a linkage— A_oA, BB_o. A_o is the center of curvature of the conchoid at its apex, and A_oA is equal to its radius of curvature. AB is made equal to the distance between focus and apex of the conchoid, and crank BB_o is made as long as possible. Link AB is extended, and the distance between B and its end-point C is made equal to the distance between focus and generating line of the conchoid. Point C then describes an approximately straight-line path in the vicinity of the apex of the conchoid, and is used as tracer point.

To summarize characteristics of the above linkages, the path of the tracer point of the symmetrical linkages, Figs 4, 6 and 7, has an infinite radius of curvature. In the linkages of Figs 1, 2, 3 and 8, the corresponding radius of curvature approaches infinity as the length of the arbitrarily assumed crank is increased. In Fig 5 the tracer of the midpoint of the coupler will have an infinite radius of curvature when the linkage is in its antisymmetrical position, provided the cranks have equal lengths.

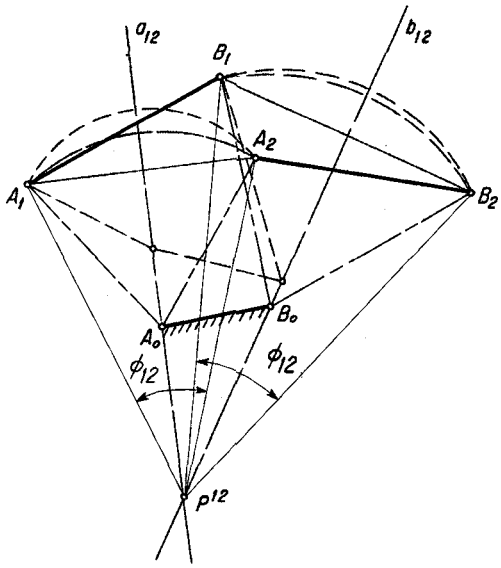


Method B

Synthesis of a Linkage Which Moves a Coupler Point Through Two Given Positions Such That Its Velocities in These Positions Are Collinear

Approach to solution: two positions of the moving crank, A_1 B_1, A_2 B_2 are arbitrarily selected; the locus of the instant center of those link configurations is constructed consistent with the velocity requirements and the selected coupler positions.

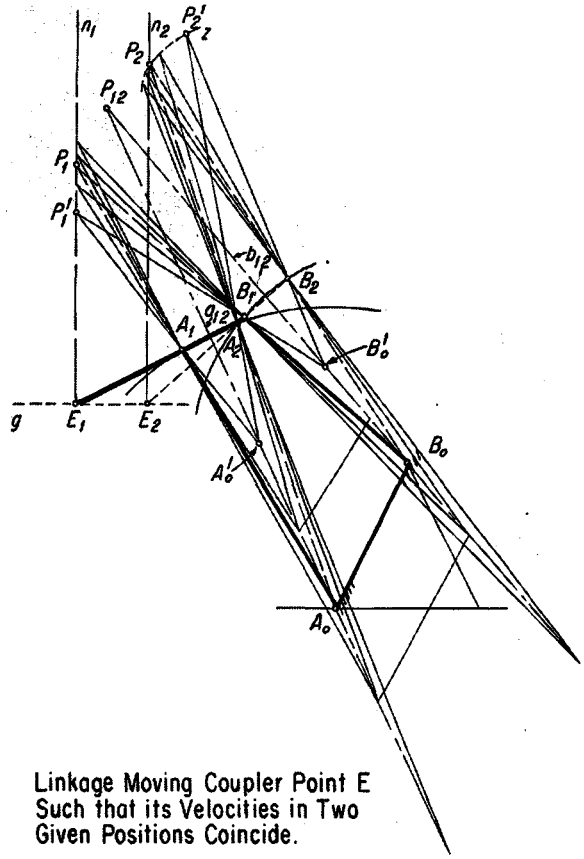
In the linkage obtained the path of the tracer point has two precision points and its tangents are collinear.



Two Coplanar Positions of A Plane System

1

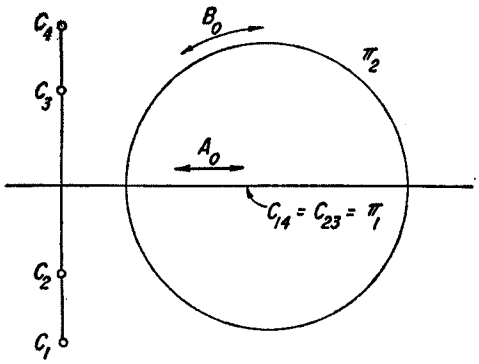
This is a preliminary explanation of motion of the essential coupler link AB before showing the method of determining the required four-bar linkage in Fig 2. Here, a plane system S, containing coupler link AB, can be moved from position A_1B_1 to coplanar position A_2B_2 by rotation through angle ϕ_{12} about pole P^{12} which is determined as the intersection of the lines of symmetry, a_{12} and b_{12}, through distances A_1—A_2 and B_1—B_2, respectively. Pole P^{12} is a point common to the two positions of AB and the reference plane. This movement of AB can be accomplished by a four-bar linkage as follows: Connect coupler point A to any pivot point A_o on line a_{12}, and coupler point B to any point B_o on line b_{12}. A_oB_o is chosen so direction of the velocities of one point of coupler AB is collinear. Thus its path approaches a straight line.



Linkage Moving Coupler Point E Such that its Velocities in Two Given Positions Coincide.

2

Two positions A_1B_1, A_2B_2 of a coupler link AB are so selected that the positions E_1, E_2 of the third point E of the coupler lie on the straight line g, representing the approximate path which point E is required to travel. This line is direction of the velocities of E in its positions E_1, E_2. The instantaneous centers, relative to the reference plane, of the coupler in positions $A_1B_1E_1$, $A_2B_2E_2$ must lie on the normals n_1 and n_2 to g through E_1 and E_2. Also, the instantaneous centers must lie on the point of intersection of the lines A_oA_1, B_oB_1, and A_oA_2, B_oB_2, respectively, where A_o and B_o denote the yet unknown crank centers. Crank center A_o must also lie on the line of symmetry a_{12} of A_1A_2, crank center B_o, on the line of symmetry b_{12} of B_1B_2. If a point P'_1 on n_1 is assumed to be the instantaneous center for link position A_1B_1, the corresponding crank centers would be A'_o, B'_o, determined by the intersection of lines P'_1A_1 with a_{12} and P'_1B_1 with b_{12} respectively. The intersection point P'_2 of the lines A'_oA_2 and B'_oB_2 determines the corresponding instantaneous center of A_2B_2. P'_2 represents one point of the locus z of all possible instantaneous centers of system position $E_2A_2B_2$ for which the instantaneous center of system position $E_1A_1B_1$ lies on line n_1. The intersection point of this curve z with n_2 gives the instantaneous center P_2 of the link position $E_2A_2B_2$, which satisfies the requirement concerning direction of velocity of point E_2 and is used to construct the desired crank centers A_o, B_o. The path of point E_1 is nearly straight for a considerable distance past the positions E_1, E_2, so that the linkage is usable through a considerable range.

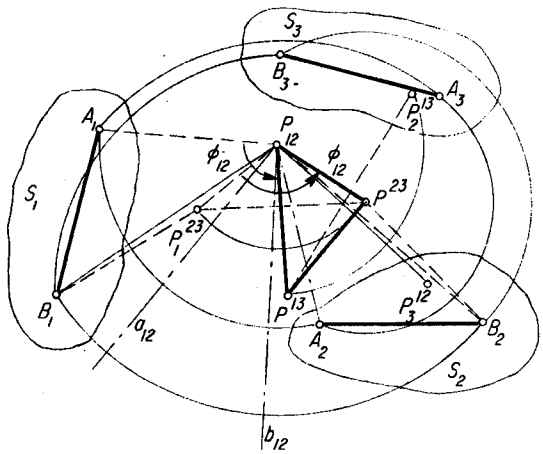


Method C

Synthesis of a Linkage by Means of Burmester's Focal Curves

Approach to solution (in its simplest form): **two pairs of positions (C_1 C_4, C_2 C_3) of a coupler point, lying on a straight line and having a common bisector C_{14} are chosen; a degenerated curve of Burmester's centers is selected so as to coincide with this bisector; the crank pivots are found by determining the poles of rotation of the coupler.**

In the linkage obtained, path of the tracer point has up to five precision points.



Three Coplanar Positions of a Plane System—the Pole Triangle

1

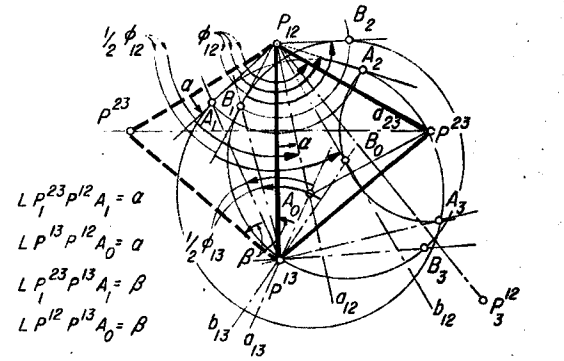
For an understanding of these methods it is first necessary to establish a certain relationship between congruent points of these coplanar positions of the plane system, their poles and angles of rotation.

For three coplanar positions S_1, S_2, S_3, there exist three poles of rotation P^{12}, P^{13}, P^{23}, forming the pole triangle which lies in the reference plane. The pairs of poles, P^{12} and P^{12}, are common to S_1S_2, S_1S_3 and the reference plane, respectively. Point P_1^{23} in the system S_1 is congruent to pole P^{23} and is found by rotating triangle $A_2B_2P^{23}$ about P^{12} until A_2B_2 coincides with A_1B_1. The angle of rotation from S_1 to S_2 is $P_1^{23}P^{12}P^{23}$, designated as ϕ_{12}.

Analogous relations apply to the points P_2^{13} congruent to P^{13} in system S_2, and P_3^{12} congruent to P^{12} in system S_3.

$\angle P_1^{23}P^{12}P^{23} = \phi_{12}$
$\angle A_1P^{12}A_2 = \phi_{12}$
$\angle A_1P^{12}A_0 = \frac{1}{2}\phi_{12}$
$\angle B_1P^{12}B_0 = \frac{1}{2}\phi_{12}$
$\angle A_1P^{12}A_0 = \angle B_1P^{12}B_0$

$\angle P_1^{23}P^{13}P^{23} = \phi_{13}$
$\angle A_1P^{13}A_3 = \phi_{12}$
$\angle A_1P^{13}A_0 = \frac{1}{2}\phi_{13}$
$\angle B_1P^{13}B_0 = \frac{1}{2}\phi_{13}$
$\angle A_1P^{13}A_0 = \angle B_1P^{13}B_0$



Three Coplanar Positions of a Plane System-relation between Pole Triangle and System Points

2

The three congruent points, A_1, A_2, A_3, lie on a circle whose center is A_0 at the intersection of lines of symmetry—a_{12}, a_{13}, and a_{23}. These lines of symmetry pass through their associated poles of rotation. Center point A_0 can be found without determining A_2 and A_3 from the noted angle relations. Angle $P_1^{23}P^{12}A_1$ must be equal to angle $P^{13}P^{12}A_0$, and angle $P_1^{23}P^{13}A_1$ must be equal to angle $P^{12}P^{13}A_0$.

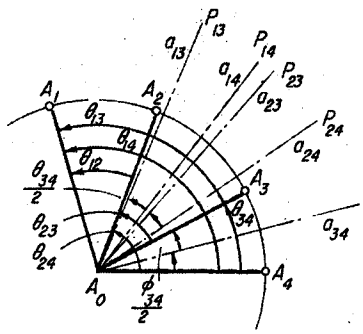
Conversely the point A_1 associated with any point A_0 can be found by the same angle relationships. Corresponding angular relations exist between B_1, B_2, B_3, and Center B_0.

$\angle A_1 A_0 P^{13} = \frac{1}{2}\theta_{13}$ $\quad$ $\angle A_2 A_0 P^{23} = \frac{1}{2}\theta_{23}$

$\angle A_1 A_0 P^{14} = \frac{1}{2}\theta_{14}$ $\quad$ $\angle A_2 A_0 P^{24} = \frac{1}{2}\theta_{24}$

$\theta_{14} - \theta_{13} = \theta_{34}$ $\quad$ $\theta_{24} - \theta_{23} = \theta_{34}$

$\angle P^{13} A_0 P^{14} = \frac{1}{2}\theta_{34}$ $\quad$ $\angle P^{23} A_0 P^{24} = \frac{1}{2}\theta_{34}$

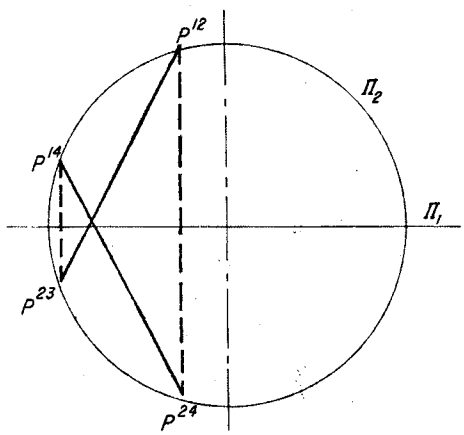


Four coplanar positions of a plane system-relation between the center point and the poles of rotation

3

To go one step further, if four coplanar positions S_1, S_2, S_3, S_4 of the moving system S are given, there exist six poles of rotation and twelve points in the system which are congruent to these poles. To determine the fixed pivots of a four-bar linkage capable of moving the system through these four system positions, two points in the moving system must be found whose congruent points in these four positions lie on a circle. Relationship between such points, A_1, A_2, A_3, A_4, and corresponding poles is shown. The center A_0 lies on the intersection of bisectors a_{13}, a_{14}, a_{23}, a_{24} of the angles $A_1A_0A_3$, $A_1A_0A_4$, $A_2A_0A_3$ and $A_2A_0A_4$. But poles P^{13}, P^{14}, P^{23} and P^{24} must also lie on these bisectors. From these angle relations it follows that the angles $P^{13}A_0P^{14} = P^{23}A_0P^{24}$, and $P^{13}A_0P^{23} = P^{14}A_0P^{24}$.

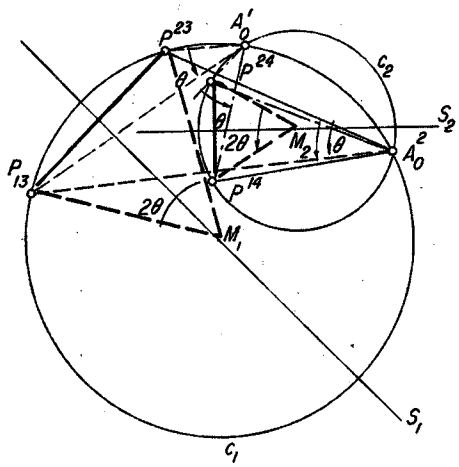
Thus, a point of reference plane at equal angles from any pair of poles with one common digit, is the center point of a circle called Burmester center—with four congruent points.



Focal curve consisting of a straight line and a curve

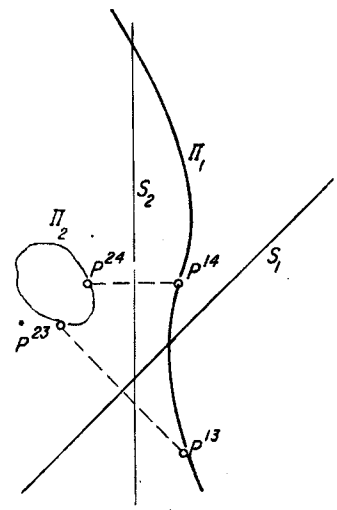
5

For certain pole positions the focal curve assumes very simple shapes. For example, if pairs of associated poles—P^{13}, P^{23}, P^{14}, P^{24}—are symmetrical to a straight line, the focal curve degenerates into straight line π_1 and circle π_2 through the four poles.



Construction of focal curve

(b)



Typical focal curve

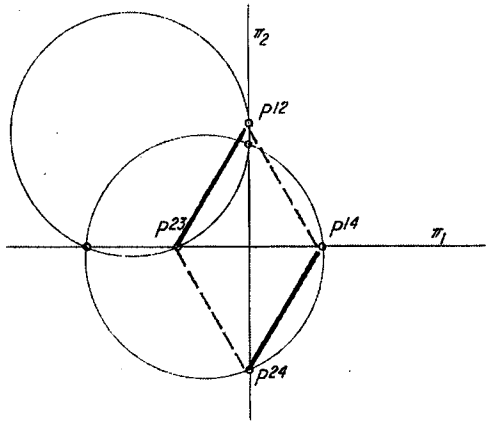
(a)

4

The locus of all such points is a plane curve of the third order, called a Focal Curve, see (a), and the Focal Curve which is the locus of all centerpoints associated with four system positions is called the centerpoint curve. It can be shown that this curve passes through all six poles, and can be constructed by selecting two pair of poles whose indices have one common digit—for example, P^{13}—P^{23}, P^{14}—P^{24}—and drawing a circle through each pair of poles such that the radii of these circles are proportional to the distance between the poles of a pair; the points of intersection of the two circles, A^1_0, A^2_0, are points of the focal curve. The radius of the crank circle associated with each center point, and the location of the congruent points can be found by the angular relations illustrated in Fig 2, p 42.

When four system positions are selected in such a manner that four congruent positions of a point of the system lie on a straight line, and for these system positions the centerpoint curve is constructed, any two points on the curve can be selected as the centers of the cranks of the desired four-bar linkage.

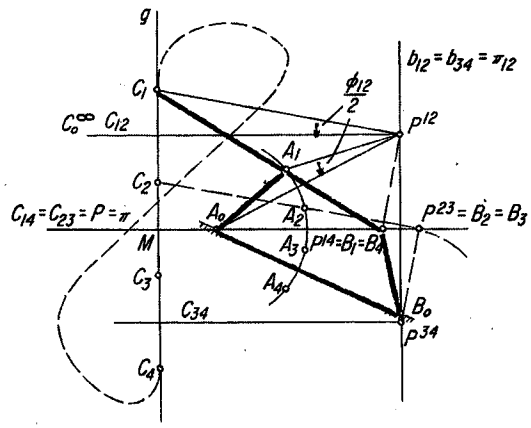
Method C continued



Focal Curve Consisting of Two Straight Lines

6

If the lines connecting associated poles constitute opposite sides of a rhombus, the focal curve degenerates into the diagonals π_1, π_2 of this rhombus and they are perpendicular to one another.

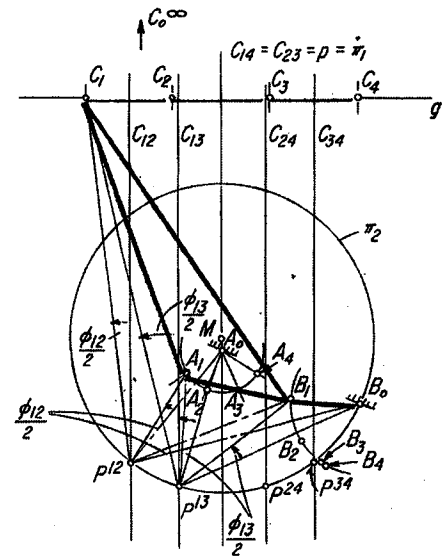


Construction of Linkage by Use of Center Point Curve Consisting of Two Straight Lines

8

This is another example of the use of the simplified shape of the centerpoint curve for the design of a linkage. As in the preceding example, four positions, C_1-C_4, C_2-C_3, on the line g of the coupler point C are selected so that they are symmetrical to the normal p. Furthermore, the positions B_1B_4 and B_2B_3 of moving point B are selected so that they coincide, respectively with one another, and also lie on line p. In this case P^{14} coincides with B_1B_4, P^{23} coincides with B_2B_3.

The poles P^{12}, P^{34} must be the intersection points of line symmetry c_{12} of C_1C_2 and c_{34} of C_3C_4 with lines of symmetry b_{12}, b_{34} of B_1B_2, B_3B_4. By choosing this configuration, pole quadrangle becomes a rhombus; consequently centerpoint curve consists of the lines of symmetry p; centerpoint B_o can be selected anywhere on b_{12}; centerpoint A_o, anywhere on p. Position A_1 of moving point A, which must lie on the line B_1C_1, can again be found by using angle relations of Fig 2, by making angle $A_1P^{12}A_o = \frac{1}{2}\phi_{12}$. Conversely, movable pivot point A may be selected anywhere on the coupler, and centerpoint A_o determined in an analogous manner.



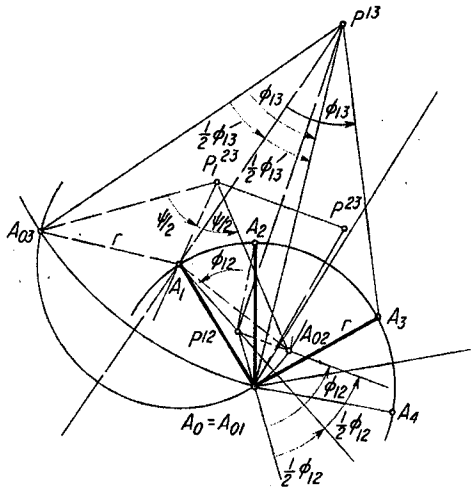
Construction of Linkage by Use of Center Point Curve Consisting of Straight Line and Circle

7

In designing a linkage by the use of the centerpoint curve it is convenient to select the system positions in such a manner that advantage is taken of the possible simplifications shown above. This is achieved here with four congruent points on a straight line chosen in such a way that C_1, C_4 and C_2, C_3 are symmetrical relative to the perpendiculars, p to g. Then the poles P^{12}, P^{13}, P^{24}, P^{34} must lie on the lines of symmetry, c_{12}, c_{13}, c_{24}, c_{34}, which are, respectfully, also symmetrical to p. Poles P^{12}, P^{13} are arbitrarily chosen on c_{12}, c_{13}, and poles P^{24}, P^{34} are placed symmetrical to them, relative to p on c_{24}, c_{34}, respectfully.

The centerpoint curve for this pole configuration is line π_1, coinciding with p, and the circle π_2, through the four poles. The crank center A_o is selected anywhere on π_1. The position A_1 of the moving pivot A is determined by the angle relations noted on Fig 2, p 42, and Fig 3, p 43.

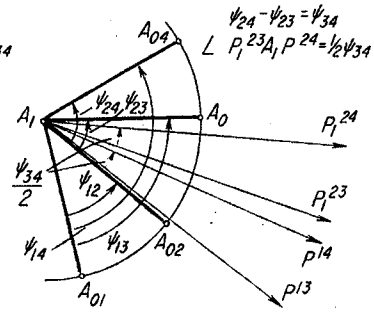
A ray through P^{12} is drawn, making the angle $C_1P^{12}C_2 = \frac{1}{2}\phi_{12}$ with line $P^{12}A_o$, and another ray is drawn through P^{13}, making the angle $C_1P^{12}C_3 = \frac{1}{2}\phi_{13}$ with line $P^{13}A_o$. The intersection point of these two rays determines the position A_1 of the moving pivot A and the length of crank A_1A_o. Crank center B_o is selected anywhere on the circle π_2, and B_1 is determined by an analogous method as was used to determine A_1.



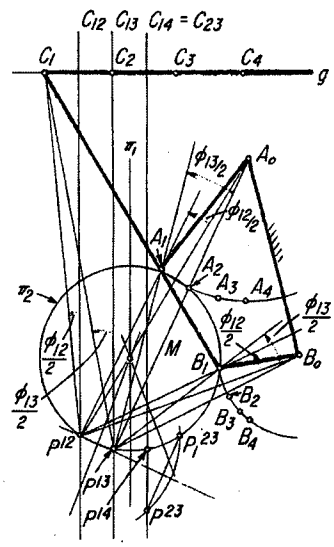
Relation Between Circle Point and Poles of Rotation

$\angle A_{01}A_1P^{13} = \frac{1}{2}\psi_{13}$
$\angle A_{01}A_1P^{14} = \frac{1}{2}\psi_{14}$
$\psi_{14} - \psi_{13} = \psi_{34}$
$\angle P^{13}A_1P^{14} = \frac{1}{2}\psi_{34}$

$\angle A_{02}A_1P_1^{23} = \frac{1}{2}\psi_{24}$
$\angle A_{02}A_1P_1^{24} = \frac{1}{2}\psi_{24}$
$\psi_{24} - \psi_{23} = \psi_{34}$
$\angle P_1^{23}A_1P^{24} = \frac{1}{2}\psi_{34}$



Four Coplanar Positions of a Plane System-Regulation between Circle Point and Poles of Rotation



Construction of Linkage by Use of Circle Point Curve Consisting of Straight Line and Circle

9

To each point of the centerpoint curve which lies in the reference plane and is the locus of all possible fixed pivots of cranks capable of moving the plane system through four prescribed coplanar positions, there is a corresponding moving pivot in the moving plane to which the crank must be attached. The relation between these points, called circle points, and the poles is illustrated here. This relation can also be used for designing linkages for generating approximately straight line motions.

Consider the systems S rotated back about the poles P^{12}, P^{13}, P^{14} to starting position S_1 from three positions, S_2, S_3, S_4. Then the points which in these system positions coincide with the center point will also rotate about the poles and since their distance from the movable pivot remains unchanged, they must also lie on a circle with the movable pivot, the circle point, as center.

This figure shows three congruent positions, A_1, A_2, A_3, of movable point A which lie on a circle with radius r, and A_o, the centerpoint, as center. Poles P^{12}, P^{13}, P^{23} are assumed to lie anywhere on the bisectors of the angles $A_1A_oA_2$, $A_1A_oA_3$, $A_2A_oA_3$. When system from position S_2 is rotated about P^{12} back into position, S_1 (held fixed in the reference plane), the point which in system position, S_2, coincides with A_o $(=A_{o1})$ will go into position A_{o2}, which is symmetrical to A_o relative to the line A_1P^{12}. When system from position S_3 is rotated back about pole P^{13}, so as also to come into coincidence with S_1, the point which in system S_3 coincided with A_o comes into position A_{o3}, which is symmetrical to A_o with respect to the line A_1P^{13}. Both points, A_{o2} and A_{o3}, lie on a circle with A_1 as center and r as radius. Pole P^{23}, through rotation about P^{12} and P^{13} respectively, goes into its congruent position P_1^{23}. Since the distance of point P_1^{23} from the points, A_{o2} and A_{o3}, is equal to distance $P^{23}A_o$, the congruent point P_1^{23}, of pole P^{23}, must lie on the line of symmetry through the points $A_{o2}A_{o3}$, which also passes through A_1.

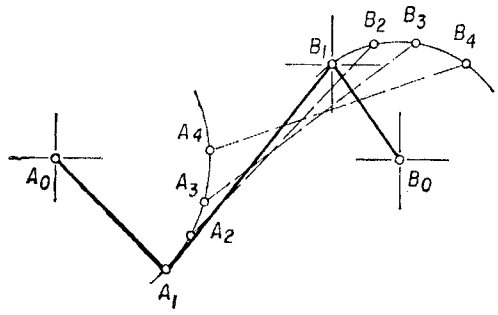
10

The locus of all circle points is also a focal curve which can be constructed in an analogous manner as the centerpoint curve, except that different pole points must be used as shown here. This relationship is analogous to Fig 3, illustrating the relationship of centerpoint to poles of rotation.

11

The circle point curve can also be used for designing a linkage for generating an approximately straight-line coupler motion. Again, by proper selection of the poles of rotation it is possible to obtain a degenerated form of the focal curve to simplify construction. Four positions of a moving system are chosen so that four congruent points of the system lie on a straight line, and their distances, C_1C_2, C_2C_3, and C_3C_4 are equal. In this case, pole P^{12} must lie on the line of symmetry c_{12} of C_1C_2; pole P^{13} on the line of symmetry c_{13} of C_1C_3 and passing through C_2, and both P^{14}, P^{23} on the line of symmetry $c_{14} = c_{23}$. For the circle point curve to degenerate into a circle and straight line (as in Fig. 5), P^{12}, P_1^{23} and P^{13}, P^{14}, respectively, must be symmetrical to a straight line. This line must be the line of symmetry π_1 of the lines c_{13},c_{14}, on which the poles P^{13}, P^{14}, respectively, lie. Therefore, P^{12} and P_1^{23} must also be symmetrical to line $P^{12}P^{13}$. If P^{12} is arbitrarily chosen on c_{12}, P_1^{23} is located symmetrical to P^{12} relative to line π_1. P^{23} is found on c_{23} by making the distance $P^{12}P^{23}$ equal to $P^{12}P_1^{23}$. Then P^{13} is determined as the point of intersection of the line of symmetry of $P^{23}P_1^{23}$ with line c_{13}, and P^{14} is the point on c_{14} which is symmetrical to p^{13} with respect to π_1. The line π_1 and the circle π_2 through $P^{12}P^{13}P^{14}P_1^{23}$ whose center, M, must also lie on π_1 are the two branches of the circle point curve of system position, $C_1A_1B_1$. The points A_1 and B_1 are chosen on π_2 so that they lie on a straight line passing through the tracer point C_1. The fixed crank centers A_o and B_o are found by considerations analogous to Fig 2, p 42. A ray is drawn through P^{12} which makes angle $\phi_{12}/2$ with A_1P^{12}. Another ray is drawn through P^{13} which makes angle $\phi_{13}/2$ with A_1P^{13}. Crank center A_o is the point of intersection of these two rays. Crank center B_o is determined in an analogous manner.

By proper selection of the positions of the poles of rotation, the circle point and centerpoint curves described above offer the simplest means for the application of Burmester's method to design of four-bar linkages.

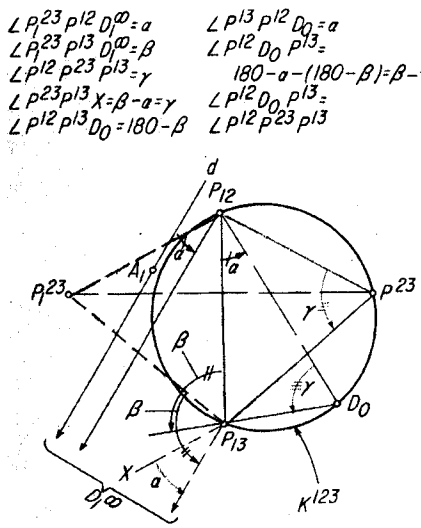


Method D

Burmester's Method for Determining, in a Given Linkage, a Coupler Point Which Performs an Approximately Straight-Line Motion

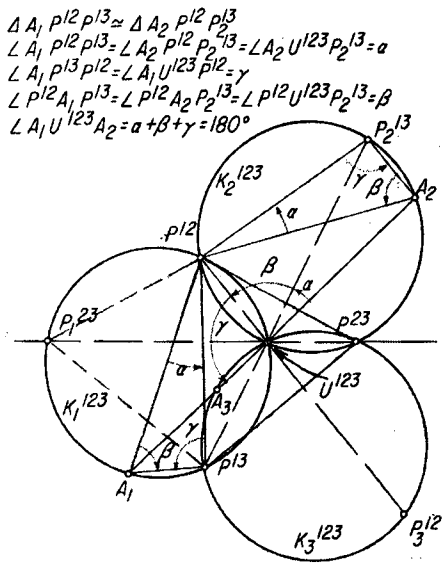
Approach to solution: four positions of a four-bar linkage ($A_1 B_1$—$A_4 B_4$) are chosen; the properties of the pole triangle and of the circle point curves of points at infinity of the coupler are used.

The path of the tracer point obtained has four precision points.


$\angle P_1^{23} P^{12} D_1^{\infty} = \alpha$
$\angle P_1^{23} P^{13} D_1^{\infty} = \beta$
$\angle P^{12} P^{23} P^{13} = \gamma$
$\angle P^{23} P^{13} X = \beta - \alpha = \gamma$
$\angle P^{12} P^{13} D_0 = 180 - \beta$
$\angle P^{13} P^{12} D_0 = \alpha$
$\angle P^{12} D_0 P^{13} =$
$180 - \alpha - (180 - \beta) = \beta - \alpha$
$\angle P^{12} D_0 D_1^{\infty} =$
$\angle P^{12} P^{23} P^{13}$
d
P_{12}
P_1^{23}
P^{23}
D_0
P_{13}
K^{123}
D_1^{∞}

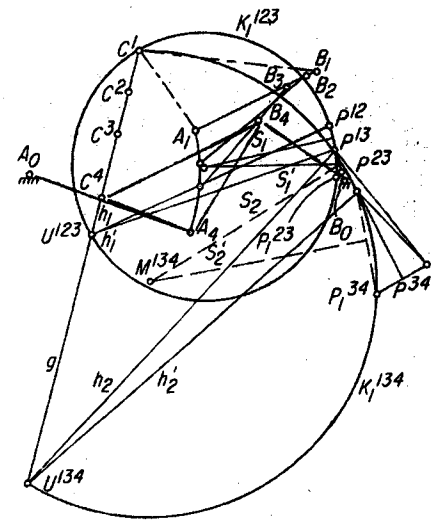
Three coplanar positions of a plane system—location of the center points of system points at infinity.

a


$\Delta A_1 P^{12} P^{13} \simeq \Delta A_2 P^{12} P_2^{13}$
$\angle A_1 P^{12} P^{13} = \angle A_2 P^{12} P_2^{13} = \angle A_2 U^{123} P_2^{13} = \alpha$
$\angle A_1 P^{13} P^{12} = \angle A_1 U^{123} P^{12} = \gamma$
$\angle P^{12} A_1 P^{13} = \angle P^{12} A_2 P_2^{13} = \angle P^{12} U^{123} P_2^{13} = \beta$
$\angle A_1 U^{123} A_2 = \alpha + \beta + \gamma = 180°$
P_2^{13}
K_2^{123}
A_2
P^{12}
P_1^{23}
P^{23}
U^{123}
A_3
K_1^{123}
P^{13}
A_1
K_3^{123}
P_3^{12}

Three coplanar positions of a plane system—location of three congruent points, relative to the pole triangle, lying on a straight line.

b



Four coplanar positions of a plane system—location of four congruent points lying on a straight line.

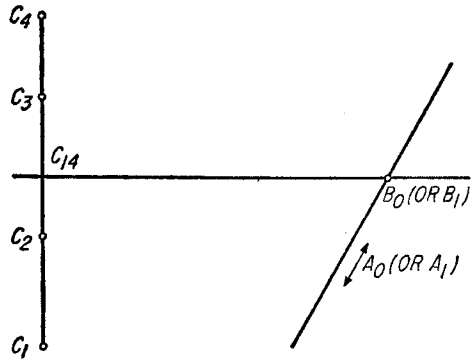
c

Since there are an infinite number of linkages capable of moving a system through four given positions in such a manner that four congruent points of the system lie on a straight line, it is possible to reverse the problem. That is, find a coupler point in a given linkage such that its four congruent positions line on a straight line. In the method developed by Burmester for solving this problem the angle relations of Fig 2, on p 42, are applied. The straight line is considered a circle with infinitely faraway center point.

If point A_1 in the system S_1—see (a) above—moves along a straight line d to infinity, then the rays $P^{12}D_1$, $P^{13}D_1$ through the infinitely faraway point D_1 at infinity become parallel. The center point D_0 associated with D_1 is constructed as in Method C, Fig 2, p 42, by making angle $P_1^{23}P^{12}D_1 =$ angle $P^{12}P^{13}D_0$, and angle $P_1^{23}P^{23}D_1 =$ angle $P^{13}P^{12}D_0$. If this procedure is applied to all infinitely faraway points of system S_1, S_2, S_3, it is evident that all center points associated with the points at infinity of the moving system lie on the circle k^{123} through $P^{12}P^{13}P^{23}$ in the reference plane.

Conversely, by the previously established to the relationship between circle point and centerpoint, the circle points associated with the center points at infinity, of the reference plane, must lie on the circles —see (b) above—k_1^{123} through $P^{12}P^{13}P_1^{23}$ in system S_1; k_2^{123} through $P^{12}P^{23}P_2^{13}$ in system S_2; k_3^{123} through $P^{13}P^{23}P_3^{12}$ in system S_3. If three congruent points, $A_1A_2A_3$, of the system lie on these circles, their centerpoint must be at infinity and the three points must therefore lie on a straight line. If two of these circles k_1^{123}, k_2^{123}, their intersection point U^{123}, and the triangles $A_1P^{12}P^{13}$, $A_2P_2^{13}P^{12}$, are considered, it is noted that the sum of the angle $A_1U^{123}P^{12}$ and $A_2U^{123}P^{12}$ is 180°, and that the straight line A_1A_2 must pass through point U^{123}. From these geometric relations it must be concluded that three circles intersect at U^{123} and that a straight line carrying three concruent points A_1, A_2, A_3 must pass through this point.

Now if four coplanar positions—see (c) —of the systems S_1, S_2, S_3, S_4, are given, the pole triangles P^{12} P^{13} P_1^{23} and P^{14} P^{24} P_1^{34} for S_1 S_2 S_3, and S_1, S_3, S_4, are determined, together with their centers of height U^{123}, U^{134}. The line connecting these two centers of height is the only line which carries four congruent positions of a point of the system. The position point, C_1, of the tracer point of the coupler is determined as the intersection point of the line U^{123}, U^{134} with the circle k_1^{123}. Center M^{123} and circle k_1^{134} center M^{134}. The linkage with the coupler point at C_4 is shown at position S_4.

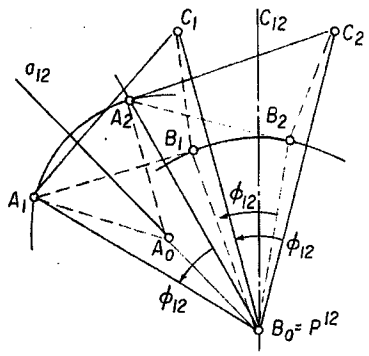


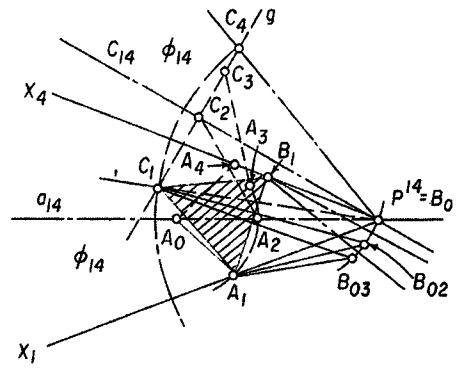
Method E

Synthesis of the Linkage by Point Position Reduction

Approach to solution (in the simplest form): four positions (C_1-C_4) of a coupler point lying on a straight line are chosen. The position of one fixed crank pivot B_0 or one position of pivot B_1 is selected on the bisector of two coupler point positions; the other pivots are determined by considering motion of selected pivot relative to coupler.

In the linkage obtained, the path of the tracer point has up to six precision points.





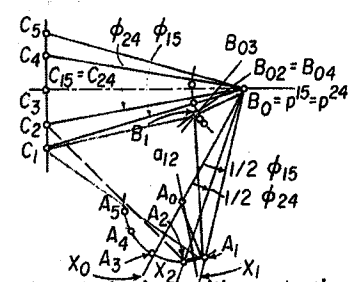
a — Method of point position reduction – one pole coinciding with fixed crank pivot, four precision points — b

1 This method, developed by Kurt Hain of the West German Federal Research Institute of Agriculture, is based on selecting the poles of rotation of the moving system in such a manner that one of them coincides with either the fixed or the moving pivot of the crank. Through this choice of one pole of rotation, relatively simple graphical solutions are possible and up to six precision points can be obtained.

This principle is illustrated in (a) above, where A_1C_1 and A_2C_2 are two positions of the line connecting the moving pivot A with the tracer point C. The fixed pivot A_0 must lie on the bisector a_{12} of A_1 and A_2; the pole P^{12} must lie on the intersection of a_{12} and the bisector c_{12} of C_1 and C_2. The fixed pivot B_0 is selected so as to coincide with P^{12}; consequently, A_0B_0 lies on a_{12}. Positions B_1 and B_2 of the second movable pivot must lie on a circle with B_0 as center. Triangles $A_1B_1C_1$ and $A_2B_2C_2$ must be congruent.

The application of the method is shown in (b) above for four points of coincidence, C_1 to C_4, with the straight line g of the path of tracer point C of the coupler. The pole P^{14} is selected anywhere on the bisector c_{14}, of the points C_1, C_4 and is assumed to coincide with the crank center B_0; then P^{14} must also lie on the bisector a_{14} of the yet unknown positions A_1, A_4 of the moving pivot A, which must also pass through the fixed pivot A_0 of the second crank.

Furthermore, angle $A_1P^{14}A_4$ must be equal to angle $C_1P^{14}C_4$ ($=\phi_{14}$), and distance A_1C_1 must be equal to A_4C_4. If two rays—X_1B_0, X_4B_0—are drawn to make angle ϕ_{14}, and an arc of any radius is drawn with B_0 as center, the intersection points A_1, A_4 of the arc with the rays will fulfill this requirement (triangles $A_1C_1B_0$ and $A_4C_4B_0$ being congruent) and also determine the distance AC. The crank positions, A_1 to A_4, must follow one another in the same sequence as the corresponding positions of point C if the path of the motion of point C between the four selected positions also must be approximately straight. To determine position of the moving pivot B, the circle point to B_0, a relationship is considered which is analogous to that shown in Fig. 9, on p 45. Consider the motion of crank BB_0 relative to A_1C_1, with the yet unknown point B_1 as fulcrum. In each of the system positions A_2C_2, A_3C_3, there is a point which coincides with B_0. If A_2C_2, A_3C_3, are rotated so as to come into coincidence with A_1C_1 these points will assume the positions B_{02}, B_{03}; these points are determined by making triangle $A_1C_1B_{02}$ congruent to triangle, $A_2C_2B_0$, and triangle $A_1C_1B_{03}$ congruent to triangle $A_3C_3B_0$. In these rotations the distances of B_{02} and B_{03} from B_1 remain equal to BB_0. This means that B_0 must lie on a circle with B_1 as center. B_1 is then found as intersection of B_0B_{02}, B_1 is then found as intersection of $B_{02}B_{03}$, $B_{02}B_{03}$, respectively.



Method of point position reduction – two poles coincide with fixed crank pivot, five precision points.

2 Since both radius and position of the fixed pivots of the second crank were arbitrarily chosen, this method can be expanded to the synthesis of linkages with five precision points. The five points shown are chosen symmetrical to point C_3 so that the lines c_{15}, c_{24} coincide with the normal through C_3 to line g. Both poles, P^{15} and P^{24}, which must lie on this line, are assumed to coincide with B_0. A ray B_0X_0 is drawn on which the yet unknown second crank center A_0 is assumed to lie. The balance of the construction is analogous to Fig 1 (b) above.

Four-Bar Linkage Proportions

Given two points, a function of the length of stroke you want, this handy chart quickly and accurately locates the other two. You find the proportions of all four-bars without cumbersome mathematical or graphical methods.

Preben W. Jensen

Proportions of 4-bar linkages for straight-line motions are usually determined by graphical methods. Then, the precision points in the required straight line are checked by mathematical methods, where necessary. But graphical methods, although simpler than a purely mathematical solution, can be quite complicated. The late Dr Kurt Rauh, professor of kinematics at the Technical Institute, Aachen, Germany, conceived the chart shown on next page, which simplifies graphical determination.

The chart is based on two circles, one having twice the diameter of the other. The smaller, called the Cardan circle, rotates inside the larger one; at the same time the center of the Cardan circle serves as a hub around which all of the space on the chart radiates.

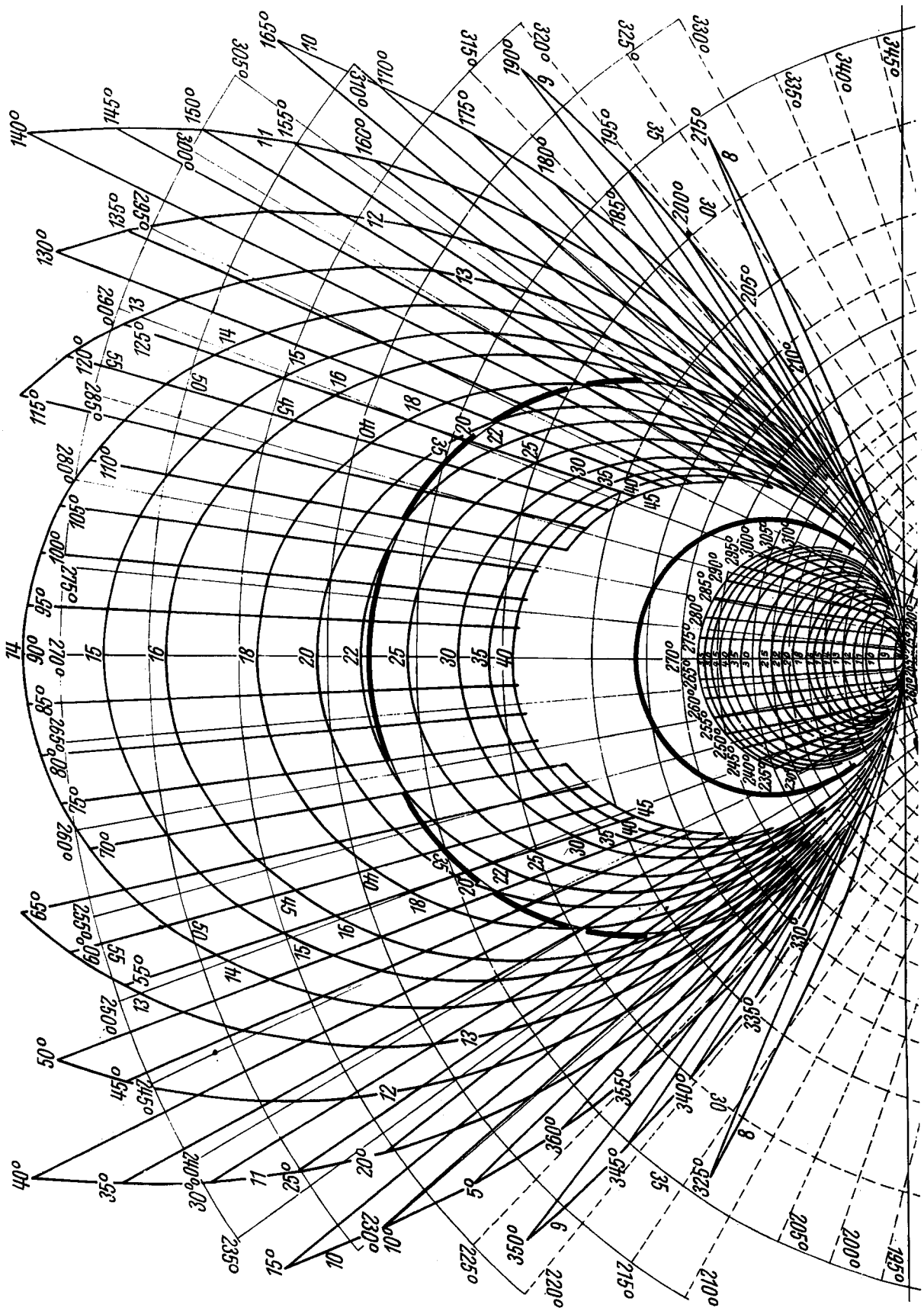
Fig 2, p 79, shows the behavior of typical points on the chart during this rotation. The center, M, of the Cardan circle, describes a circular path around center, W, of the large circle; all points on the circumference of the Cardan circle describe straight-line paths at different angles through W. All other points, whether inside or outside the Cardan circle, describe elliptical paths.

The diagram shows the elliptical paths of two arbitrary points, E_1 and E_2. Portions of these elliptical paths have been replaced by circular arcs, and the centers of curvature, K_1 and K_2, found. Note that these centers are on an extension of the lines PE_1 and PE_2, where P is the common pole of the two circles.

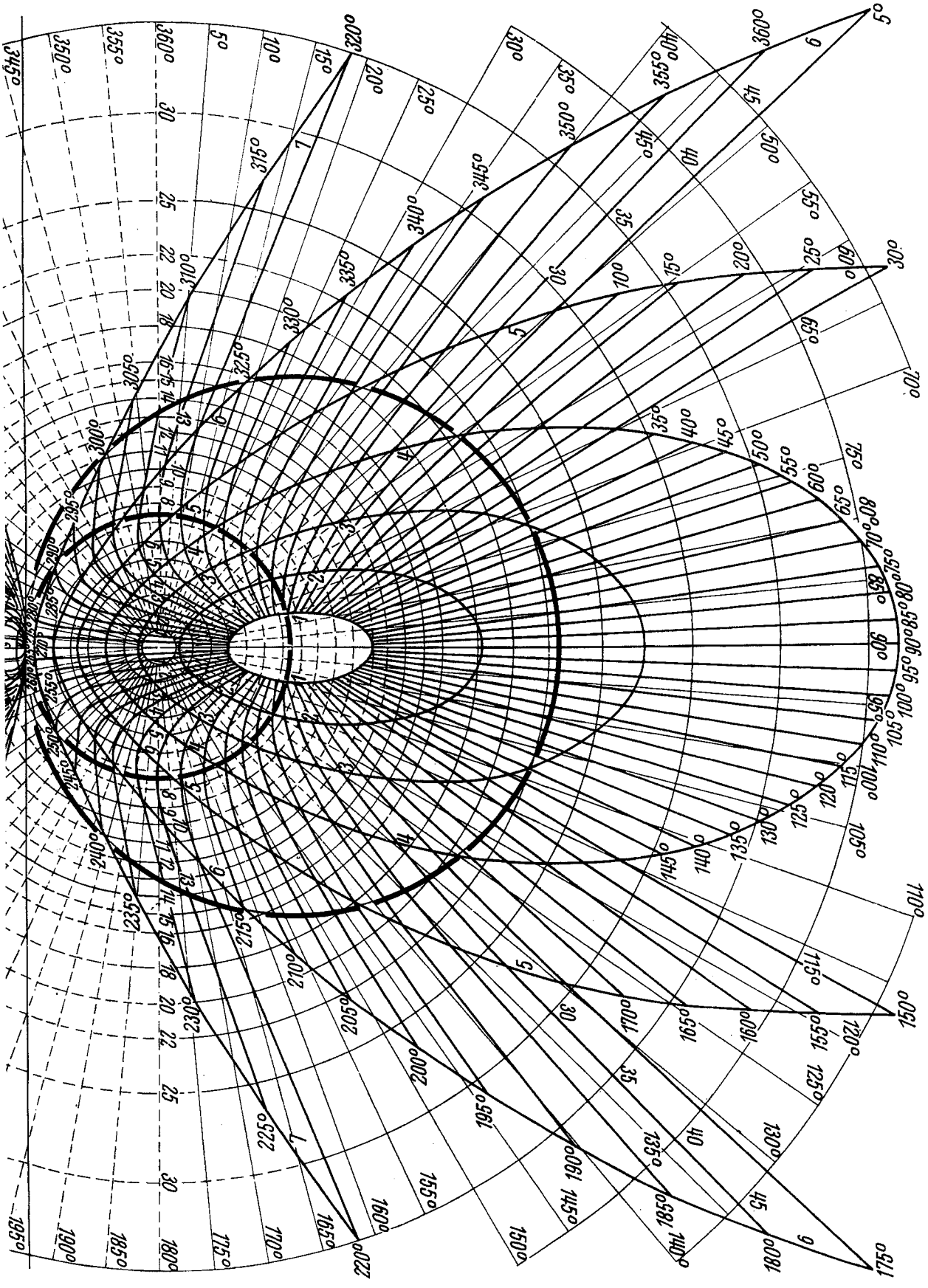
If E_1K_1 and E_2K_2 are taken as two links in a 4-bar linkage with K_1 and K_2 as stationary points, points E_1 and E_2 will be the coupler points describing circular arcs which, within a certain range, approximate the elliptical paths.

If these coupler points are connected to W on the circumference of the Cardan circle, this point will describe the desired straight line as long as points E_1 and E_2 move on those parts of the circular arcs that

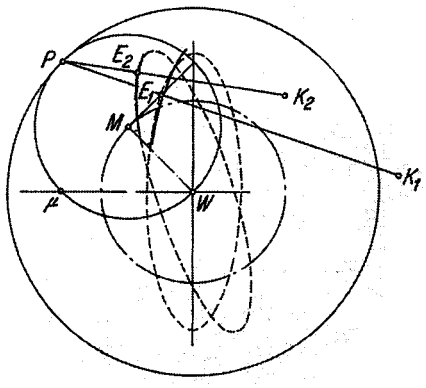
1 . . PROPORTIONS OF 4-BAR LINKAGES are determined with this special chart—made up of two separate systems. One finds coupler points; other finds stationary points. For coupler points, read figures from circles and rays. For stationary points, read figures from ovals and curved rays.



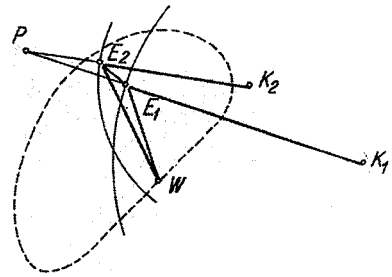
For coupler points, turn this way



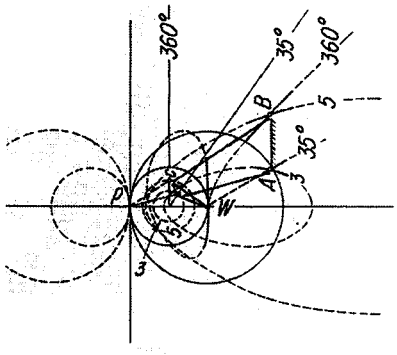
For stationary points, turn this way



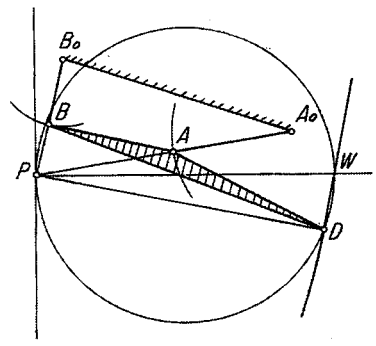
2 . . **PATHS OF TYPICAL POINTS** during rotation of Cardan circle. Here, circular arcs have been drawn replacing parts of the elliptical paths of E_1 and E_2. K_1 and K_2 are centers of curvature of these arcs.

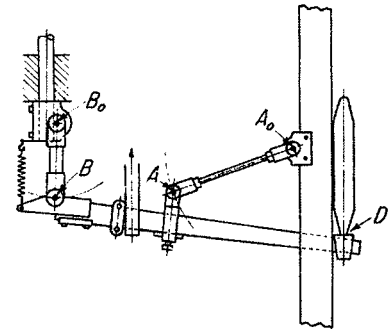


3 . . **ACTUAL LINKAGE** determined in Fig 2 causes point W to describe the desired straight-line path during part of its complete path which is shown by dotted line.



4 . . **POINTS FOUND** from special chart produce desired straight-line motion. Points A and B are arbitrary; points D and C determined from chart.





5 & 6 . . **WEAVING MECHANISM** (right) is designed to fit points (left) determined by the chart. A_o and B_o are chosen to give required length of travel for point D; A and B are found from chart. Note that D travels at slight angle from vertical.

closely approximate the elliptical paths they replace. The complete path described by coupler point W is shown dotted in Fig 3 together with the 4-bar linkage that produces it. This linkage results from the arbitrary selection of points E_1 and E_2 and the development of centers K_1 and K_2 from the chart.

How the Chart Works

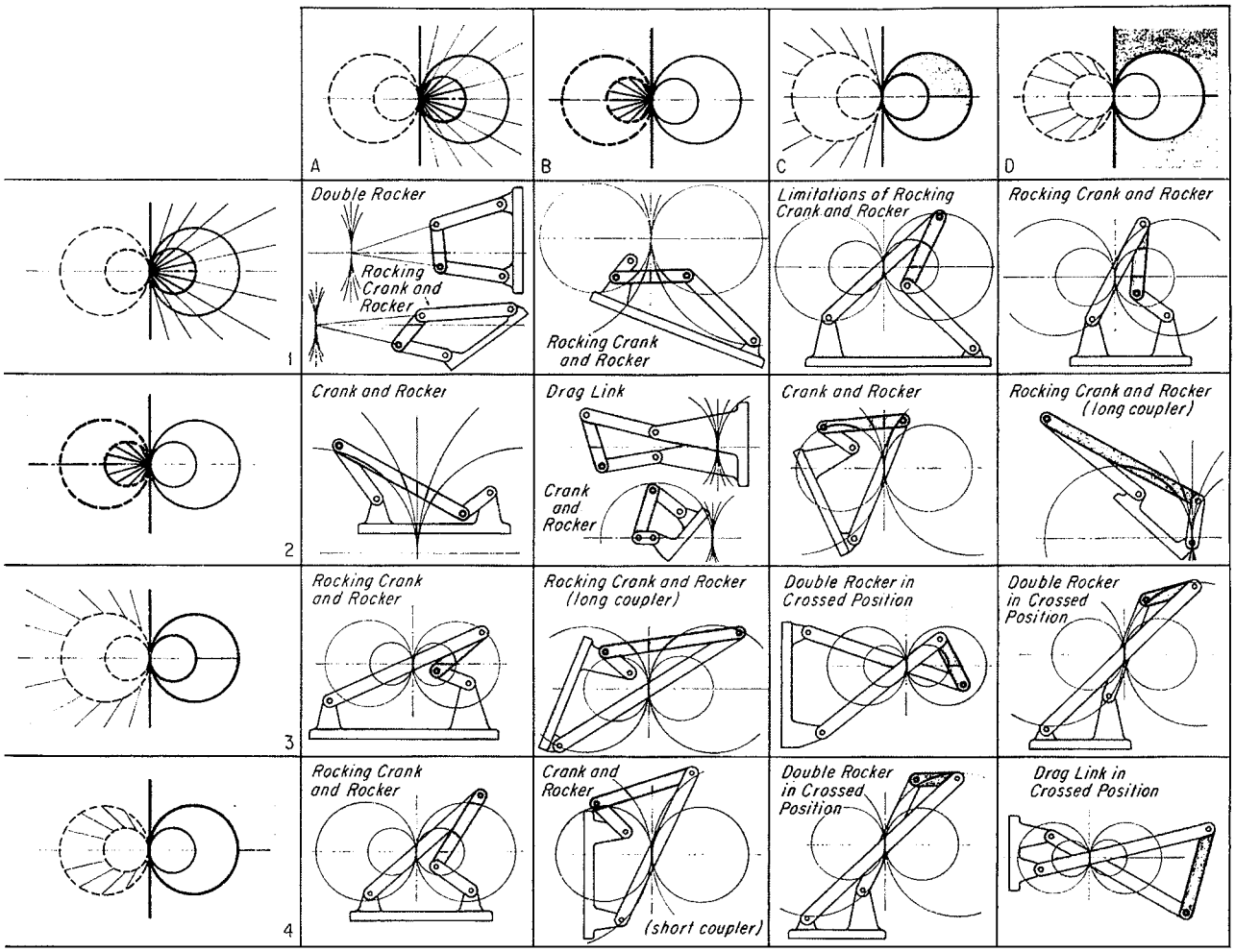
The chart has been constructed so that you can find the coupler points corresponding to two given stationary points (centers of curvature) or vice versa. Here is how:

Concentric with the Cardan circle is a system of circles that are divided by rays at 5° intervals. The intersections represent possible coupler points. A companion system of ovals, divided by curved rays at 5° intervals, represents the possible fixed points (centers of curvature). To use the chart, start with whichever pair of points, coupler or fixed, are predetermined. Then locate these points in the appropriate system; the corresponding points will be located at the same-numbered intersections in the other system. When the coupler points thus determined are connected to a point on the Cardan circle, this point will produce straight-line motion through point W.

Solving a Sample Problem

In Fig 4, straight-line motion is desired at point W. Stationary points A and B are selected as shown. A falls on the intersections of oval 3 and curved ray 35°; thus corresponding coupler point D is at the intersection of circle 3 and ray 35°. Similarly, B falls at intersection of oval 5 and curved ray 360°; its coupler point, C, at intersection of circle 5 and ray 360°.

Coupler points D and C are connected to W on the circumference of the Cardan circle, which will then describe the desired straight line as shown. The best approximation of a straight line is obtained when the coupler points are connected to W or points near it. The straight-line portion of the path described by W is in a vertical direction. When C and D are connected to other points on the Cardan circle, the straight-line portion will go through W at an angle to the vertical, depending upon distance of the chosen point



7 . . A VARIETY OF MECHANISMS can be produced by choosing points from different parts of the chart. In each case, colored coupler is produced when one point is taken from colored area in first column and other from colored area in top row.

from W. Thus any desired direction of travel is possible in theory. However, there is this limitation: the further the chosen point is from W, the less accurate the approximation of straight-line motion.

This effect is more clearly shown in Fig 5, which gives another 4-bar linkage proportioned from the chart. In this case points A_0 and B_0 are centers of curvature for coupler points A and B, which are connected to tracer point D. A_0 and B_0 are predetermined points on the machine; A and B are found from the chart.

The straight-line portion of the coupler path described by point D goes through W at an angle as shown. A typical application of this 4-bar linkage is the weaving mechanism (Fig 6), used in the textile industry. Proportions for this mechanism are determined from the sample 4-bar linkage determined in Fig 5.

Wide Variety Possible

Coupler points for a 4-bar linkage can fall anywhere on the chart, depending upon the type of mechanism required. Fig 7 shows diagrammatically the wide variety of mechanisms that may be determined from the chart The relation of the coupler points to their correspond ing centers of rotation to the Cardan circle determines the type of mechanism. For example, if both coupler points fall within the Cardan circle on the right-hand side of the chart, the type of mechanism will be 1A. On the other hand, if both coupler points fall on the left-hand side of the chart, the type of mechanism will be 2B. However, any combination of these is possible, as can be seen from the diagrams.